

Pasting pseudofunctors

Suresh Nayak

ABSTRACT. We give an abstract criterion for pasting pseudofunctors on two subcategories of a category into a pseudofunctor on the whole category. As an application we extend the variance theory of the twisted inverse image $(-)^!$ over schemes to that over formal schemes. Thus we show that over composites of compactifiable maps of noetherian formal schemes, there is a pseudofunctor $(-)^!$ such that if f is a pseudoproper map, then $f^!$ is the right adjoint to the derived direct image $\mathbf{R}f_*$ and if f is étale, then $f^!$ is the inverse image f^* . We also show that $(-)^!$ is compatible with flat base change.

CONTENTS

1. Introduction	1
2. The abstract pasting results	5
3. Proofs I (generalized isomorphisms in the labeled setup)	13
4. Proofs II (the cocycle condition)	31
5. Proofs III (old isomorphisms and linearity)	47
6. Proofs IV (the output)	57
7. Applications	65
References	77

1. Introduction

1.1. Background. The purpose of this paper is to give an abstract criterion for pasting pseudofunctors on two subcategories of a category into a pseudofunctor on the whole category. The inspiration for the criterion comes from applications lying in the theory of Grothendieck duality.

For the sake of this introduction, let \mathbb{S} be the category of separated finite-type maps of noetherian schemes and \mathbb{F} the category of separated pseudofinite-type maps of noetherian formal schemes. Here \mathbb{S} can be thought of as the full subcategory of \mathbb{F} consisting of formal schemes whose structure sheaf of rings has discrete topology.

The author thanks the Mathematisches Forschungsinstitut Oberwolfach, the Banff International Research Station and the Institute of Mathematical Sciences at Chennai for providing access to their conducive environment in which much of this research was carried out. The author was funded by the National Board of Higher Mathematics.

One of the main applications of our pasting result concerns extending the variance theory of the twisted-inverse-image functor $(-)^!$ to the setup of formal schemes. Recall that, in [9, Theorem 2] it has been shown that for any map $f: \mathcal{X} \rightarrow \mathcal{Y}$ in \mathbb{F} , the functor $\mathbf{R}f_*: \mathbf{D}_{\text{qct}}^+(\mathcal{X}) \rightarrow \mathbf{D}_{\text{qct}}^+(\mathcal{Y})$ has a right adjoint f^\times , where $\mathbf{D}_{\text{qct}}^+(-)$ stands for the derived category of bounded-below complexes whose homology modules are quasi-coherent and torsion. Let $(-)^*$ be the resulting pseudofunctor on \mathbb{F} . Following the case of ordinary schemes, a basic goal in the abstract side of duality theory over formal schemes is of constructing a pseudofunctor $(-)^!$, possibly over the whole of \mathbb{F} , such that the restriction of $(-)^!$ to the subcategory of pseudoproper maps is isomorphic to $(-)^*$, and the restriction to open immersions is the inverse-image pseudofunctor $(-)^*$. Thus the problem of constructing $(-)^!$ is a problem of pasting pseudofunctors.

It is known that a solution for this pasting problem exists over \mathbb{S} , henceforth to be also called as the classical case, though a complete presentation of details has not appeared in published form yet. But there have been serious hurdles in carrying over the approach of the classical case to that over \mathbb{F} . Let us first briefly recall the main inputs used in the classical case.

Over \mathbb{S} , the main ingredients used for pasting are (i) Nagata's compactification theorem ([7], [8]) which says that every map f in \mathbb{S} factors as $f = pi$ where p is a proper map and i an open immersion; (ii) The open-base-change theorem due to Deligne and more generally, the flat-base-change theorem of Verdier ([11]). While (ii) has been generalized to the formal-scheme setup ([9, Theorem 3]), we do not know if the analog of (i) for formal schemes holds.

The nonavailability of (i) has been a critical hurdle for pasting over \mathbb{F} . In fact, pasting even within the limited scope of compactifiable maps in \mathbb{F} had not been achieved since the classical proofs, in their intermediate stages, rely on some form of (i). For instance, in the situation of a map f in \mathbb{F} admitting two factorizations $f = p_1 i_1 = p_2 i_2$ where p_j are pseudoproper and i_j open immersions, a canonical choice for an isomorphism $i_1^* p_1^\times \xrightarrow{\sim} i_2^* p_2^\times$ had not been demonstrated till now. (see [10])

The functorial details that need to be addressed in these pasting problems are best tackled in an abstract framework. In [1] Deligne gave an abstract criterion for pasting pseudofunctors. But its requirement that factorizations such as in (i) above hold in the working category is a serious drawback. Moreover, even in some contexts where (i) does hold, satisfying Deligne's input conditions seems to require considerable functorial manipulations which could possibly be carried out at an abstract level itself.

Against this backdrop we offer a way of pasting that does not rely on the existence of factorizations as in (i). Instead, our approach exploits compatibilities concerning what we call fundamental isomorphisms.

In the context of pasting $(-)^*$ with $(-)^*$, a fundamental isomorphism is one associated to every factorization of an identity map $1_X = gf$ with g open and f pseudoproper, and is of the form $f^\times g^* \xrightarrow{\sim} \mathbf{1}_{\mathbf{D}_{\text{qct}}^+(\mathcal{X})}$. Such an isomorphism is easily constructed: in this case f, g are necessarily isomorphisms and hence we may use $g^* = g^\times$ and $f^\times g^\times \xrightarrow{\sim} (1_X)^\times$. Nevertheless, these fundamental isomorphisms play a very useful role in pasting.

Our principal results are abstract in nature in the spirit of Deligne’s pasting result. Before summarizing, we point out the main application to duality that comes out of them.

Loosely speaking, we now have the following result (see Theorems 7.1.3, 7.1.4 and 7.1.6).

Over the subcategory of composites of compactifiable maps in \mathbb{F} (equivalently, the subcategory of composites of open immersions and pseudoproper maps in \mathbb{F}) there is a pseudofunctor $(-)^!$ whose restriction to the subcategory of pseudoproper maps is isomorphic to $(-)^{\times}$ and whose restriction to the subcategory of open immersions is $(-)^$, and which furthermore, is uniquely determined via these isomorphisms by its compatibility with open-base-change isomorphisms. Moreover $(-)^!$ satisfies compatibility with flat base change and can also be extended to the subcategory of composites of étale maps and pseudoproper maps in \mathbb{F} .*

Note however that since we do not know whether the subcategory of composites of compactifiable maps in \mathbb{F} (or even composites of étale and pseudoproper) is the whole of \mathbb{F} , therefore we do not know if $(-)^!$ as constructed by us is defined over the whole of \mathbb{F} .

Finally, here is a rough summary of our abstract pasting result. (Theorem 2.2.4)

For its input, we assume that there is a category \mathcal{C} and there are pseudofunctors $(-)^{\times}$ and $(-)^{\square}$ defined over two subcategories \mathcal{P} and \mathcal{O} respectively. Furthermore, we assume that to every fibered square \mathfrak{s} resulting from the fibered product of a \mathcal{P} -map with an \mathcal{O} -map there is an associated “base-change” isomorphism $\beta_{\mathfrak{s}}$ and that to every factorization σ of an identity map into a \mathcal{P} -map followed by an \mathcal{O} -map, there is an associated “fundamental” isomorphism ϕ_{σ} . The comparison maps of $(-)^{\times}, (-)^{\square}$, the base-change isomorphisms β_{-} and the fundamental isomorphisms ϕ_{-} are assumed to be compatible in certain ways.

For the output we obtain a pseudofunctor $(-)^!$ on the smallest subcategory \mathcal{Q} of \mathcal{C} containing \mathcal{P} and \mathcal{O} , together with isomorphisms between the restrictions of $(-)^!$ to \mathcal{P}, \mathcal{O} and $(-)^{\times}, (-)^{\square}$ respectively such that $(-)^!$ is compatible with β_{-} and ϕ_{-} .

We also have an abstract analog for flat base change. (Theorem 2.3.2)

1.2. Localness on source of torsion twisted inverse image. In order to illustrate some of our main arguments and concerns, we take up here the following task. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a map in \mathbb{F} and let $\mathcal{X} \xrightarrow{i} \mathcal{Z}_1 \xrightarrow{p} \mathcal{Y}$ and $\mathcal{X} \xrightarrow{j} \mathcal{Z}_2 \xrightarrow{q} \mathcal{Y}$ be factorizations of f such that p, q are pseudoproper and i, j are open immersions. Let us call these factorizations as $\sigma_1 := (i, p)$ and $\sigma_2 := (j, q)$. Our aim is to give a canonical isomorphism $c(\sigma_2, \sigma_1): i^* p^{\times} \xrightarrow{\sim} j^* q^{\times}$.

Consider the following diagram where the squares involved are fiber-product diagrams.

$$\begin{array}{ccccccc}
 \mathcal{X} & \xrightarrow{\Delta} & \mathcal{W}_{11} & \xrightarrow{i_2} & \mathcal{W}_{12} & \xrightarrow{p_2} & \mathcal{X} \\
 & & j_2 \downarrow & & j_1 \downarrow & & \downarrow j \\
 & & \mathcal{W}_{21} & \xrightarrow{i_1} & \mathcal{W}_{22} & \xrightarrow{p_1} & \mathcal{Z}_2 \\
 & & q_2 \downarrow & & q_1 \downarrow & & \downarrow q \\
 & & \mathcal{X} & \xrightarrow{i} & \mathcal{Z}_1 & \xrightarrow{p} & \mathcal{Y}
 \end{array}$$

We claim that there are natural isomorphisms

$$(\dagger) \quad \Delta^\times j_2^* q_2^\times \xrightarrow{\sim} \mathbf{1} \xleftarrow{\sim} \Delta^\times i_2^* p_2^\times$$

where $\mathbf{1}$ is the identity on $\mathbf{D}_{\text{qct}}^+(\mathcal{X})$. Assuming the claim we proceed as follows. Consider isomorphisms β_i for $i = 1, 2, 3, 4$ as shown below where β_1 and β_4 are the obvious pseudofunctorial isomorphisms, and β_2, β_3 result from the base-change isomorphism [9, Theorem 3] corresponding to the northeastern and southwestern squares above respectively.

$$j_2^* i_1^* \xrightarrow{\beta_1} i_2^* j_1^* \quad j_1^* p_1^\times \xrightarrow{\beta_2} p_2^\times j^* \quad q_2^\times i^* \xrightarrow{\beta_3} i_1^* q_1^\times \quad q_1^\times p^\times \xrightarrow{\beta_4} p_1^\times q^\times$$

Then the desired isomorphism $c(\sigma_2, \sigma_1)$ is obtained via the following ones

$$\begin{aligned} i^* p^\times &\xleftarrow{\sim} \Delta^\times j_2^* q_2^\times i^* p^\times \xrightarrow{\text{via } \beta_3} \Delta^\times j_2^* i_1^* q_1^\times p^\times \xrightarrow{\text{via } \beta_1} \Delta^\times i_2^* j_1^* q_1^\times p^\times \\ &\xrightarrow{\text{via } \beta_4} \Delta^\times i_2^* j_1^* p_1^\times q^\times \\ &\xrightarrow{\text{via } \beta_2} \Delta^\times i_2^* p_2^\times j^* q^\times \xrightarrow{\sim} j^* q^\times. \end{aligned}$$

For the isomorphisms in (\dagger) , let us begin with the first one, viz., $\Delta^\times j_2^* q_2^\times \xrightarrow{\sim} \mathbf{1}$. Consider the following diagram described below.

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & & \downarrow a_1 & & \\ & & \mathcal{V} & \xrightarrow{a_2} & \mathcal{X} \\ & & \downarrow a_3 & & \downarrow a_4 \\ \mathcal{X} & \xrightarrow[\Delta]{} & \mathcal{W}_{11} & \xrightarrow{j_2} & \mathcal{W}_{21} \xrightarrow{q_2} \mathcal{X} \end{array}$$

Here $a_4 := j_2 \Delta$, the square is obtained by taking fibered products and a_1 is the diagonal map. Thus a_1, a_3, a_4 are all pseudoproper, in fact closed immersions. In particular, via pseudofunctoriality we obtain isomorphisms

$$\Delta^\times \xrightarrow{\sim} a_1^\times a_3^\times, \quad a_4^\times q_2^\times \xrightarrow{\sim} \mathbf{1}.$$

Since $a_2 a_1 = 1_{\mathcal{X}}$, it follows that a_2 is surjective. Since a_2 is also an open immersion, therefore it is an isomorphism. In particular, a_2^* and $a_{2*} = \mathbf{R}a_{2*}$ are both left-adjoint and right-adjoint to each other. Hence we may use $a_2^* = a_2^\times$. There results a natural isomorphism

$$a_1^\times a_2^* = a_1^\times a_2^\times \xrightarrow{\sim} (1_{\mathcal{X}})^\times = \mathbf{1}.$$

We define the first isomorphism in (\dagger) via the following isomorphisms where $\beta_5: a_3^\times j_2^* \xrightarrow{\sim} a_2^* a_4^\times$ is the inverse of the base-change isomorphism associated to the fibered square in the preceding diagram.

$$\Delta^\times j_2^* q_2^\times \xrightarrow{\sim} a_1^\times a_3^\times j_2^* q_2^\times \xrightarrow{\text{via } \beta_5} a_1^\times a_2^* a_4^\times q_2^\times \xrightarrow{\sim} a_4^\times q_2^\times \xrightarrow{\sim} \mathbf{1}.$$

The second isomorphism in (\dagger) is obtained through a similar procedure.

Thus we have demonstrated an isomorphism $c(\sigma_2, \sigma_1): i^* p^\times \xrightarrow{\sim} j^* q^\times$. In what sense is it canonical? We discuss some points concerning this.

To begin with, note the usage of fiber-product diagrams in the construction. Since fibered products are unique only up to isomorphism, the objects $\mathcal{W}_{ij}, \mathcal{V}$ used above and the arrows coming in and out of them are not unique as functions of the

maps that we started with. Therefore we must verify that $c(\sigma_2, \sigma_1)$ is independent of the choices of these objects. This can be done by comparing two different choices of fiber-product diagrams via the unique isomorphisms relating them.

Clearly $c(\sigma_2, \sigma_1)$ is a natural transformation, even one of triangulated functors.

Finally, if there is a third factorization σ_3 of f , then the cocycle condition, viz., $c(\sigma_1, \sigma_2) \circ c(\sigma_2, \sigma_3) = c(\sigma_1, \sigma_3)$ holds. Proving this requires working with triple-fiber-product diagrams involving the three factorizations.

Since we achieve canonicity at the level of complexes, our result is an improvement upon the one in [10, Prop. 2], where the isomorphism obtained is canonical only at the level of the homology modules.

Much of our work is concentrated around generalizing the isomorphism $c(\sigma_1, \sigma_2)$ and its cocycle property to the situation of factorizations of arbitrary length, i.e., the case when σ_1 and σ_2 are arbitrarily long sequences of open and pseudoproper maps. Addressing the concerns raised above then requires considerable effort. The whole task is carried out in an abstract setting.

* * *

In §2 we state the abstract results without proof. The proofs span §3–§6. In §7 we state some applications, most of which concern Grothendieck duality.

1.3. Conventions.

- (i) We interchangeably use the terms cartesian square and fibered square. We call a diagram composed of cartesian squares as a cartesian diagram.
- (ii) Reduced notation : Given functors

$$\Gamma_1: \mathcal{B} \rightarrow \mathcal{A}, \quad \Gamma_2: \mathcal{C} \rightarrow \mathcal{B}, \quad \Gamma'_2: \mathcal{C} \rightarrow \mathcal{B}, \quad \Gamma_3: \mathcal{D} \rightarrow \mathcal{C},$$

and a natural transformation $\theta: \Gamma_2 \rightarrow \Gamma'_2$, we call a natural transformation $\delta: \Gamma_1\Gamma_2\Gamma_3 \rightarrow \Gamma_1\Gamma'_2\Gamma_3$ as “ θ under reduced notation” if δ is the obvious transformation induced by θ and acting as identity on Γ_1, Γ_3 . Here $\Gamma_1, \Gamma_2, \Gamma'_2, \Gamma_3$ may occur as composites of other functors too.

2. The abstract pasting results

Here we present the abstract pasting results without proofs. The basic input data for pasting is stated in §2.1 below while the output occurs in Theorem 2.2.4. In §2.3 we discuss the abstract form of flat-base-change results. The proofs of all these results span §3–§6.

We begin with some preliminaries.

Recall that a *contravariant normalized pseudofunctor* $(-)^{\#}$ on a category \mathbf{C} assigns to every object X in \mathbf{C} a category $X^{\#}$, to every map $f: X \rightarrow Y$ in \mathbf{C} a functor $f^{\#}: Y^{\#} \rightarrow X^{\#}$, and to every pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbf{C} an isomorphism $C_{f,g}^{\#}: f^{\#}g^{\#} \xrightarrow{\sim} (gf)^{\#}$ such that the following conditions hold:

- $C_{-, -}^{\#}$ is associative vis-à-vis triple compositions.
- $(1_Z)^{\#} = 1_{Z^{\#}}$ for all objects Z in \mathbf{C} .
- For any $f: X \rightarrow Y$ in \mathbf{C} the following isomorphisms are identity

$$f^{\#} = 1_X^{\#} f^{\#} \xrightarrow{C_{1_X, f}^{\#}} f^{\#}, \quad f^{\#} = f^{\#} 1_Y^{\#} \xrightarrow{C_{f, 1_Y}^{\#}} f^{\#}.$$

Unless mentioned otherwise, any pseudofunctor occurring in this paper shall be assumed to be contravariant and normalized.

A morphism of pseudofunctors $(-)^{\#} \rightarrow (-)^!$ on \mathbf{C} consists of the following data:

- For every object X in \mathbf{C} , there is a functor $S_X: X^{\#} \rightarrow X^!$;
- For every morphism $f: X \rightarrow Y$ in \mathbf{C} there is a natural transformation $S_X f^{\#} \rightarrow f^! S_Y$;

such that for any pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following diagram (of obvious natural maps) commutes,

$$\begin{array}{ccccc} S_X f^{\#} g^{\#} & \longrightarrow & f^! S_Y g^{\#} & \longrightarrow & f^! g^! S_Z \\ \downarrow & & & & \downarrow \\ S_X (gf)^{\#} & \longrightarrow & (gf)^! S_Z & & \end{array}$$

and for any object X , the natural map $S_X = S_X 1_X^{\#} \rightarrow 1_X^! S_X = S_X$ is the identity.

The composition of two maps of pseudofunctors $(-)^{\#} \rightarrow (-)^! \rightarrow (-)^*$ is defined in the obvious way. Note that if $(-)^{\#} \rightarrow (-)^!$ is an isomorphism, then the associated functors S_X and natural transformations $S_X f^{\#} \rightarrow f^! S_Y$ are isomorphisms.

2.1. The input data for gluing. Here then is the input data for gluing, consisting of [A]-[D] below.

[A]. There is a category \mathbf{C} and there are subcategories \mathbf{O}, \mathbf{P} that are stable under base change by maps in \mathbf{C} , i.e., for any map $f: X \rightarrow Y$ in \mathbf{P} (resp. \mathbf{O}) and any map $g: Z \rightarrow Y$ in \mathbf{C} , the fibered product of f with g exists and the induced map $f': X' \rightarrow Z$ is also in \mathbf{P} (resp. \mathbf{O}). Moreover we require that the following holds.

- The category \mathbf{I} of isomorphisms in \mathbf{C} is contained in \mathbf{O} and in \mathbf{P} . (In particular, every object of \mathbf{C} is also in \mathbf{O} and \mathbf{P} .)
- Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in \mathbf{C} such that gf is in \mathbf{P} . If g is in \mathbf{O} or in \mathbf{P} , then f is in \mathbf{P} . (Therefore, the conclusion that f is in \mathbf{P} also holds if g is assumed to be a composite of maps in \mathbf{O} and \mathbf{P} .)

[B]. There is a pseudofunctor $(-)^{\times}$ on \mathbf{P} and a pseudofunctor $(-)^{\square}$ on \mathbf{O} such that $X^{\times} = X^{\square}$ for any object X in \mathbf{C} . Henceforth we use \mathcal{D}_X for X^{\times} or X^{\square} .

[C]. For any cartesian square \mathfrak{s} in \mathbf{C} as follows such that f is in \mathbf{P} and i is in \mathbf{O} (so that f' is in \mathbf{P} and i' in \mathbf{O}),

$$\begin{array}{ccc} U & \xrightarrow{i'} & X \\ f' \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

there is a “base-change” isomorphism $\beta_{\mathfrak{s}}: i'^{\square} f^{\times} \xrightarrow{\sim} f'^{\times} i^{\square}$, also sometimes referred to as $\beta_{f,i}$ is there is no cause for confusion. Moreover β_{-} obeys the transitivity rules (i) and (ii) stated below.

Consider the following extensions of \mathfrak{s} where $j \in \mathbf{O}$ and $g \in \mathbf{P}$. In each case the appended square is also cartesian and is called \mathfrak{s}_1 , while the composite cartesian

square is called \mathfrak{c} .

$$\begin{array}{ccccc}
 & & & U_1 & \xrightarrow{i''} & X_1 \\
 & & & g' \downarrow & & g \downarrow \\
 U_1 & \xrightarrow{j'} & U & \xrightarrow{i'} & X \\
 f'' \downarrow & & f' \downarrow & & f \downarrow \\
 V_1 & \xrightarrow{j} & V & \xrightarrow{i} & Y \\
 & & & f' \downarrow & & f \downarrow \\
 & & & V & \xrightarrow{i} & Y
 \end{array}$$

(i) Horizontal transitivity: The following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
 j' \square i' \square f^\times & \xrightarrow{j' \square (\beta_s)} & j' \square f' \times i' \square & \xrightarrow{\beta_{s1}(i' \square)} & f'' \times j' \square i' \square \\
 C_{j',i'}^\square(f^\times) \downarrow & & & & \downarrow f'' \times (C_{j,i}^\square) \\
 (i' j')^\square f^\times & \xrightarrow{\beta_c} & & & f'' \times (ij)^\square
 \end{array}$$

(ii) Vertical transitivity: The following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
 i'' \square g^\times f^\times & \xrightarrow{\beta_{s1}(f^\times)} & g' \times i' \square f^\times & \xrightarrow{g' \times (\beta_s)} & g' \times f' \times i' \square \\
 i'' \square (C_{g,f}^\times) \downarrow & & & & \downarrow C_{g',f'}^\times(i' \square) \\
 i'' \square (gf)^\times & \xrightarrow{\beta_c} & & & i' \square (g' f')^\times
 \end{array}$$

[D]. For any object X in \mathbf{C} and for any sequence $X \xrightarrow{f} Y \xrightarrow{g} X$ factoring the identity map on X such that $g \in \mathbf{O}$ (so that f is necessarily in \mathbf{P}), there is a “fundamental” isomorphism $\phi_{f,g}: f^\times g^\square \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$. Moreover $\phi_{f,g}$ is compatible with isomorphisms defined earlier in the following ways (i) and (ii).

(i) Let $h: X' \rightarrow X$ be a map in \mathbf{C} . Consider the following induced diagram where $g' f' = 1_{X'}$ and each square is cartesian, with the one on the left being called \mathfrak{l} and the one on right \mathfrak{r} .

$$\begin{array}{ccccc}
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & X' \\
 h \downarrow & & h' \downarrow & & h \downarrow \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & X
 \end{array}$$

Then the following conditions hold.

(a) If h is in \mathbf{P} then the following diagram of isomorphisms commutes.

$$\begin{array}{ccccc}
 f' \times g' \square h^\times & \xrightarrow{\text{via } \beta_r} & f' \times h' \times g^\square & \xrightarrow{\text{via } (-)^\times} & h^\times f^\times g^\square \\
 \phi_{f',g'}(h^\times) \downarrow & & & & \downarrow h^\times(\phi_{f,g}) \\
 \mathbf{1}_{\mathcal{D}_{X'}} h^\times & \xlongequal{\quad} & h^\times & \xlongequal{\quad} & h^\times \mathbf{1}_{\mathcal{D}_X}
 \end{array}$$

- (b) If h is in \mathbf{O} then the following diagram of isomorphisms commutes where \mathfrak{l}^t is the transpose of \mathfrak{l} .

$$\begin{array}{ccccc}
 f' \times g' \times h^\square & \xrightarrow{\text{via } (-)^\square} & f' \times h' \times g^\square & \xrightarrow{\text{via } \beta_{\mathfrak{l}^t}^{-1}} & h^\square f \times g^\square \\
 \phi_{f',g'}(h^\square) \downarrow & & & & \downarrow h^\square(\phi_{f,g}) \\
 \mathbf{1}_{\mathcal{D}_X}, h^\square & \xlongequal{\quad} & h^\square & \xlongequal{\quad} & h^\square \mathbf{1}_{\mathcal{D}_X}
 \end{array}$$

- (ii) For any diagram as follows where the square \mathfrak{s} is cartesian, $f, h, i \in \mathbf{P}$, $g, j, k \in \mathbf{O}$ and $gf = 1_X = ki$,

$$\begin{array}{ccccc}
 & X & & & \\
 f \downarrow & & & & \\
 Y & \xrightarrow{g} & X & & \\
 h \downarrow & \mathfrak{s} & \downarrow i & & \\
 Z & \xrightarrow{j} & W & \xrightarrow{k} & X
 \end{array}$$

the following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
 f \times h \times j^\square k^\square & \xrightarrow{\text{via } \beta_{\mathfrak{s}}} & f \times g^\square i \times k^\square \\
 \text{via } C_{f,h}^\times \text{ and } C_{j,k}^\square \downarrow & & \downarrow \text{via } \phi_{f,g} \text{ and } \phi_{i,k} \\
 (hf)^\times (kj)^\square & \xrightarrow{\phi_{hf,kj}} & \mathbf{1}_{\mathcal{D}_X}
 \end{array}$$

2.2. The output. It will be convenient to develop some terminology before stating the output corresponding to the input conditions of §2.1.

Let $\mathbf{Q} = \{\mathbf{O}, \mathbf{P}\}$ be the smallest subcategory of \mathbf{C} containing \mathbf{O} and \mathbf{P} . Then \mathbf{Q} is also the category, whose objects are all the objects of \mathbf{C} and whose morphisms are composites of maps in \mathbf{O} and \mathbf{P} .

DEFINITION 2.2.1. A *pseudofunctorial cover* \mathcal{C} (with respect to data $[A]$ – $[D]$) is a triple $((-)^!, \mu_\times^!, \mu_\square^!)$ consisting of

- (i) a pseudofunctor $(-)^!$ on \mathbf{Q} ;
- (ii) a pseudofunctorial morphism $\mu_\times^! : (-)^!|_{\mathbf{P}} \rightarrow (-)^\times$ on \mathbf{P} ;
- (iii) a pseudofunctorial morphism $\mu_\square^! : (-)^!|_{\mathbf{O}} \rightarrow (-)^\square$ on \mathbf{O} ;

subject to the following conditions (a), (b) and (c).

- (a) For any object $X \in \mathbf{C}$, the functors $X^! \rightarrow X^\times = \mathcal{D}_X$ and $X^! \rightarrow X^\square = \mathcal{D}_X$ induced by $\mu_\times^!$ and $\mu_\square^!$ respectively, are the same. Let us denote this functor $X^! \rightarrow \mathcal{D}_X$ by S_X .
- (b) For every cartesian square \mathfrak{s} in \mathbf{C} as follows such that $f \in \mathbf{P}$ and $i \in \mathbf{O}$,

$$\begin{array}{ccc}
 U & \xrightarrow{i'} & X \\
 f' \downarrow & & \downarrow f \\
 V & \xrightarrow{i} & Y
 \end{array}$$

the following diagram commutes.

$$\begin{array}{ccc}
S_U i'^! f^! & \xrightarrow{\text{via } (-)^!} & S_U f'^! i^! \\
\text{via (iii)} \downarrow & & \downarrow \text{via (ii)} \\
i'^\square S_X f^! & & f'^\times S_Y i^! \\
\text{via (ii)} \downarrow & & \downarrow \text{via (iii)} \\
i'^\square f^\times S_Y & \xrightarrow{\text{via } \beta_\mathfrak{s}} & f'^\times i^\square S_Y
\end{array}$$

- (c) For every sequence $X \xrightarrow{i} Y \xrightarrow{h} X$ such that $hi = 1_X, i \in \mathbf{P}, h \in \mathbf{O}$, the following diagram commutes.

$$\begin{array}{ccc}
S_X i^! h^! & \xrightarrow{\text{via } (-)^!} & S_X \mathbf{1}_X^! \\
\text{via (ii)} \downarrow & & \parallel \\
i^\times S_Y h^! & & S_X \\
\text{via (iii)} \downarrow & & \parallel \\
i^\times h^\square S_X & \xrightarrow{\text{via } \phi_{i,h}} & \mathbf{1}_{\mathcal{D}_X} S_X
\end{array}$$

2.2.2. A morphism between two pseudofunctorial covers is defined as follows. Let $\mathcal{C}_1 = ((-)^!, \mu_\times^!, \mu_\square^!)$, $\mathcal{C}_2 = ((-)^#, \mu_\times^\#, \mu_\square^\#)$ be two pseudofunctorial covers. Then a morphism $\mathcal{C}_2 \rightarrow \mathcal{C}_1$ of pseudofunctorial covers is a morphism of pseudofunctors $\epsilon: (-)^\# \rightarrow (-)^!$ such that the following two diagrams commute.

$$\begin{array}{ccc}
(-)^\#|_{\mathbf{P}} & \xrightarrow{\epsilon|_{\mathbf{P}}} & (-)^!|_{\mathbf{P}} \\
& \searrow & \swarrow \\
& & (-)^\times
\end{array}
\quad
\begin{array}{ccc}
(-)^\#|_{\mathbf{O}} & \xrightarrow{\epsilon|_{\mathbf{O}}} & (-)^!|_{\mathbf{O}} \\
& \searrow & \swarrow \\
& & (-)^\square
\end{array}$$

DEFINITION 2.2.3. A pseudofunctorial cover $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ is called perfect if $\mu_\times^!, \mu_\square^!$ are isomorphisms.

The output for the input data of §2.1 may now be stated as follows. For convenience we call a pseudofunctorial cover simply as a cover.

THEOREM 2.2.4. *Under input conditions [A]–[D] of §2.1 the following hold.*

- (i) *There exists a perfect cover.*
- (ii) *Any perfect cover is final in the category of all covers, i.e., for any perfect cover \mathcal{C} and any cover \mathcal{C}' there exists a unique map of covers $\mathcal{C}' \rightarrow \mathcal{C}$. In particular, any two perfect covers are isomorphic via a unique isomorphism.*

A proof of this theorem is given in §6.2 and is based on the results of §3–5.

REMARK 2.2.5. Suppose we further assume for the input conditions that for any object X in \mathbf{C} , \mathcal{D}_X is a triangulated category and that the functors f^\times, f^\square for f in \mathbf{P} or \mathbf{O} are triangulated functors that “commute” with translation and that $C_{-, -}^\times, C_{-, -}^\square, \beta_-$ and ϕ_- are morphisms of triangulated functors. Then the perfect

cover of Theorem 2.2.4, say $\mathcal{C} = ((-)^!, \mu_{\times}^!, \mu_{\square}^!)$, can be chosen such that for any f in \mathcal{Q} , $f^!$ is a triangulated functor that commutes with translation and such that the isomorphisms $C_{-,-}^!, \mu_{\times}^!, \mu_{\square}^!$ respect the appropriate triangulated structures. This will follow from the proof of 2.2.4.

REMARK 2.2.6. A curious consequence of Theorem 2.2.4(i) is that on the intersection \mathbf{R} of \mathbf{P} and \mathbf{O} , the restrictions of $(-)^{\times}$ and $(-)^{\square}$ to \mathbf{R} are isomorphic as pseudofunctors. This is clearly a necessary condition for pasting but we do not find it useful to have it as part of the input data. In particular, it is not a substitute for the more powerful condition [D]. (cf. 7.1.6 where [D] is easy to attain.)

For now, we give a quick indication of how to construct, for any map $f: X \rightarrow Y$ in \mathbf{R} , a natural isomorphism $f^{\square} \xrightarrow{\sim} f^{\times}$. Consider the following diagram where the square \mathfrak{s} is cartesian and Δ is the diagonal map, which, by §2.1[A](i),(ii), is in \mathbf{P} .

$$\begin{array}{ccccc} X & \xrightarrow{\Delta \in \mathbf{P}} & X^2 & \xrightarrow{\pi_2 \in \mathbf{O}} & X \\ & & \downarrow \pi_1 \in \mathbf{P} & \mathfrak{s} & \downarrow f \in \mathbf{P} \\ & & X & \xrightarrow{f \in \mathbf{O}} & Y \end{array}$$

Then the desired isomorphism is obtained as

$$f^{\square} \xrightarrow{\text{via } (1_X)^{\times} \cong \Delta^{\times} \pi_1^{\times}} \Delta^{\times} \pi_1^{\times} f^{\square} \xrightarrow{\text{via } \beta_{\mathfrak{s}}} \Delta^{\times} \pi_2^{\square} f^{\times} \xrightarrow{\text{via } \phi_{\Delta, \pi_2}} f^{\times}.$$

Thus for obtaining this isomorphism alone, the compatibilities in [C], [D] of the input data are not needed. This isomorphism is generalized further in 4.1.3 below. Verdier's proof of the dualizing property of differentials ([11, Theorem 3]) can be seen to be an instance of this by using $\mathbf{O} = \text{smooth maps}$, etc..

2.3. Flat base change. Now we consider change of base by maps in a new subcategory of \mathbf{C} and a corresponding new class of base-change isomorphisms. We distinguish it from the base-change isomorphisms of §2.1[C], by calling it *flat* base change.

2.3.1. Consider the following addition [E1]–[E3] to the input data [A]–[D] of §2.1.

[E1]. There is a subcategory \mathbf{F} of \mathbf{C} that is stable under base change by maps in \mathbf{O} or \mathbf{P} and contains the subcategory \mathbf{I} of isomorphisms in \mathbf{C} .

[E2]. On \mathbf{F} there is a pseudofunctor $(-)^{\flat}$ such that $X^{\flat} = \mathcal{D}_X$ for any object X in \mathbf{C} .

[E3]. Let \mathfrak{s} be a cartesian square as follows where f is in \mathbf{O} or \mathbf{P} and u is in \mathbf{F} .

$$\begin{array}{ccc} W & \xrightarrow{u'} & X \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{u} & Y \end{array}$$

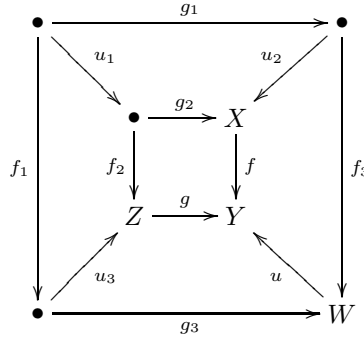
Then the following conditions (i) and (ii) hold subject to (a) and (b) below.

- (i) If f is in \mathbf{P} , then there is an isomorphism $\beta_{\mathfrak{s}}^{\times}: u'^{\flat} f^{\times} \xrightarrow{\sim} f'^{\times} u^{\flat}$, also sometimes written as $\beta_{f,u}^{\times}$ if there is no cause for confusion. Moreover β_{-}^{\times} is transitive vis-à-vis extensions of \mathfrak{s} via \mathbf{F} -maps horizontally or \mathbf{P} -maps vertically.

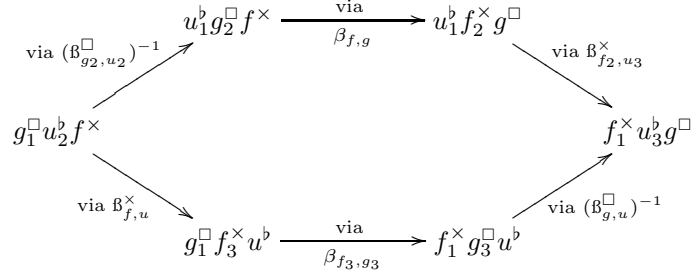
- (ii) If f is in \mathbf{O} , then there is an isomorphism $\beta_{\mathfrak{s}}^{\square}: u^b f^{\square} \xrightarrow{\sim} f'^{\square} u^b$, also sometimes written as $\beta_{f,u}^{\square}$ if there is no cause for confusion. Moreover β_{-}^{\square} is transitive vis-à-vis extensions of \mathfrak{s} via \mathbf{F} -maps horizontally or \mathbf{O} -maps vertically.

The “flat-base-change” isomorphisms $\beta_{-}^{\times}, \beta_{-}^{\square}$ are compatible with the isomorphisms β_{-} and $\phi_{-,-}$ of §2.1[C] and §2.1[D] as follows.

- (a) For any cartesian “cube” as follows, (i.e., each “face” of the cube is a fibered product diagram)



if $f \in \mathbf{P}, g \in \mathbf{O}, u \in \mathbf{F}$, then the following hexagon of isomorphisms commutes.



- (b) For any cartesian diagram as follows where $gf = 1_X$, $f \in \mathbf{P}, g \in \mathbf{O}, u \in \mathbf{F}$,

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & X' \\ u \downarrow & & u' \downarrow & & u \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & X \end{array}$$

the following diagram of isomorphisms commutes.

$$\begin{array}{ccccc} f'^{\times} g'^{\square} u^b & \xrightarrow[\beta_{g,u}^{\square})^{-1}]{\text{via}} & f'^{\times} u'^b g^{\square} & \xrightarrow[\beta_{f,u'}^{\square})^{-1}]{\text{via}} & u^b f^{\times} g^{\square} \\ \text{via } \phi_{f',g'} \downarrow & & & & \downarrow \text{via } \phi_{f,g} \\ \mathbf{1}_{\mathcal{D}_{X'}} u^b & \equiv & u^b & \equiv & u^b \mathbf{1}_{\mathcal{D}_X} \end{array}$$

Under the conditions [A]–[D], [E1]–[E3], we obtain the following output.

THEOREM 2.3.2. *Let $\mathcal{C} = ((-)^!, \mu_{\times}^!, \mu_{\square}^!)$ be a perfect pseudofunctorial cover. Then there is a unique family of flat-base-change isomorphisms $\beta_{-}^!$, (i.e., for any*

cartesian square \mathfrak{s} as follows where $f \in \mathbf{Q} = \overline{\{\mathbf{O}, \mathbf{P}\}}$ and $u \in \mathbf{F}$,

$$\begin{array}{ccc} W & \xrightarrow{u'} & X \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{u} & Y \end{array}$$

there is an isomorphism $\mathcal{B}_s^!: u'^b f^! \xrightarrow{\sim} f'^! u^b$ such that the following hold.

- (i) $\mathcal{B}_s^!$ is vertically and horizontally transitive.
- (ii) For \mathfrak{s} as above, if f is in \mathbf{P} (resp. in \mathbf{O}), then the following diagram on the left (resp. on the right) commutes.

$$\begin{array}{ccc} u'^b f^! & \xrightarrow{\mathcal{B}_s^!} & f'^! u^b \\ \text{via } \mu_{\times}^! \downarrow & & \downarrow \text{via } \mu_{\times}^! \\ u'^b f^{\times} & \xrightarrow{\beta_s^{\times}} & f'^{\times} u^b \end{array} \quad \begin{array}{ccc} u'^b f^! & \xrightarrow{\mathcal{B}_s^!} & f'^! u^b \\ \text{via } \mu_{\square}^! \downarrow & & \downarrow \text{via } \mu_{\square}^! \\ u'^b f^{\square} & \xrightarrow{\beta_s^{\square}} & f'^{\square} u^b \end{array}$$

In particular, if $\mathcal{C}' = ((-)^{\#}, \mu_{\times}^{\#}, \mu_{\square}^{\#})$ is another perfect cover with $\mathcal{B}_s^{\#}$ the corresponding family of flat-base-change isomorphisms, then for \mathfrak{s} as above, via the unique isomorphism of covers $\mathcal{C}' \rightarrow \mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} u'^b f^{\#} & \xrightarrow{\mathcal{B}_s^{\#}} & f'^{\#} u^b \\ \downarrow & & \downarrow \\ u'^b f^! & \xrightarrow{\mathcal{B}_s^!} & f'^! u^b \end{array}$$

A proof of this theorem is given in §6.3.

2.4. A variant. We give an alternate criterion for pasting pseudofunctors and the related base-change result.

2.4.1. For input data, we assume that the following conditions (1)–(5) hold.

- (1) Conditions [A], [B] of §2.1 and [E1], [E2], [E3](i) of §2.3 hold.
- (2) The subcategory \mathbf{O} is contained in \mathbf{F} and $(-)^{\square}$ is the restriction of $(-)^b$ to \mathbf{O} . For a fibered square \mathfrak{s} resulting from the change of base of a \mathbf{P} -map by an \mathbf{O} -map, we set $\beta_s := \beta_s^{\times}$.
- (3) If i is an isomorphism in \mathbf{C} , then $i^{\square} = i^{\times}$.
- (4) If $X \xrightarrow{f} Y \xrightarrow{g} X$ are \mathbf{C} -maps such that $gf = 1_X, f \in \mathbf{P}, g \in \mathbf{O}$, then f, g are isomorphisms.
- (5) Let \mathfrak{s} be a fibered square as follows where $f, g \in \mathbf{P}$ and $i, j \in \mathbf{F}$.

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

If i, j are isomorphisms, then among the following diagrams, the left one commutes while if f, g are isomorphisms, then the right one commutes.

$$\begin{array}{ccc}
 j^\square f^\times & \xrightarrow{\beta_s} & g^\times i^\square \\
 \parallel & & \parallel \\
 j^\times f^\times & \xrightarrow{\text{via } (-)^\times} & g^\times i^\times
 \end{array}
 \qquad
 \begin{array}{ccc}
 j^b f^\times & \xrightarrow{\beta_s^\times} & g^\times i^b \\
 \parallel & & \parallel \\
 j^b f^b & \xrightarrow{\text{via } (-)^b} & g^b i^b
 \end{array}$$

2.4.2. We define a pseudofunctorial cover \mathcal{C} (with respect to data (1)–(5)) the same way as in 2.2.1, except for the following modification. We do not require 2.2.1(c) to hold, but instead require that for every isomorphism $i: X \rightarrow Y$ in \mathcal{C} , the natural maps $S_X i^! \rightarrow i^\times S_Y$ and $S_X i^! \rightarrow i^\square S_Y$ are the same via $i^\times = i^\square$.

The notion of a morphism of covers and that of a perfect cover is the same as before.

For a fibered square \mathfrak{s} resulting from the change of base of an \mathcal{O} -map by an \mathcal{F} -map, we set β_s^\square to be the obvious map resulting from pseudofunctoriality of $(-)^\square$.

THEOREM 2.4.3. *Under the input conditions (1)–(5), the following hold.*

- (i) *There exists a perfect cover. Any such cover is final in the category of all covers.*
- (ii) *For any perfect cover, there is a unique family of flat-base-change isomorphisms such that the conclusion of 2.3.2 holds.*

We give a proof of this theorem in §6.4.

3. Proofs I (generalized isomorphisms in the labeled setup)

The proofs of the output statements of §2.2, §2.3 and §2.4 are based on the results of the next few sections. The general theme underlying these proofs is of working with sequences of (composable) \mathcal{C} -maps where each map is in \mathcal{O} or in \mathcal{P} . After proving results at the level of such sequences, one then descends to statements concerning maps in $\mathcal{Q} = \{\mathcal{O}, \mathcal{P}\}$.

In this section we introduce the setup of labeled maps and sequences, and corresponding diagrams. This is the setup in which the results of §4 and §5 are stated and proved. The notion of labeled maps provides us with a uniform way of treating similar results through common notation, thus making the transition to working with sequences easier. Most of our work in this section goes into showing that the base-change isomorphisms and the fundamental isomorphisms of the input data of §2.1 can be upgraded to the level of labeled sequences.

3.1. Labeled maps and sequences. For a map f which is both, in \mathcal{O} and in \mathcal{P} , the input conditions of §2.1 provide us with two immediate choices for a functor, viz., f^\square and f^\times . This motivates the following definitions as a way of keeping track of the choice whenever necessary.

DEFINITION 3.1.1. A *labeled map* in \mathcal{C} is a pair $F := (f, \lambda)$ where f is a map in \mathcal{O} or \mathcal{P} and λ , the label, is an element of the set $\{\mathcal{O}, \mathcal{P}\}$ such that $f \in \lambda$.

DEFINITION 3.1.2. For a labeled map $X \xrightarrow{F = (f, \lambda)} Y$ in \mathcal{C} we define a functor from \mathcal{D}_Y to \mathcal{D}_X by

$$F^\boxtimes = (f, \lambda)^\boxtimes := \begin{cases} f^\square, & \text{if } \lambda = \mathcal{O}; \\ f^\times, & \text{if } \lambda = \mathcal{P}. \end{cases}$$

DEFINITION 3.1.3. By a labeled sequence $\sigma := F_1, \dots, F_n$ we mean a sequence of labeled maps

$$X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} X_3 \longrightarrow \dots \longrightarrow X_n \xrightarrow{F_n} X_{n+1}.$$

Suppose $F_i = (f_i, \lambda_i)$. We define $|\sigma|$ to be the composite map $f_n \cdots f_1: X_1 \rightarrow X_{n+1}$. We call X_1 the source of σ , X_{n+1} its target and n its length. We frequently also use double-arrow notation such as $\sigma: X_1 \Longrightarrow X_{n+1}$ to denote a labeled sequence.

DEFINITION 3.1.4. For any labeled sequence $\sigma = F_1, \dots, F_n$ we define the functor $\sigma^\boxtimes: \mathcal{D}_{X_{n+1}} \rightarrow \mathcal{D}_{X_1}$ by $\sigma^\boxtimes := F_1^\boxtimes F_2^\boxtimes \cdots F_n^\boxtimes$.

One of the main constructions used in the proofs of the output statements in §2.2 and §2.3 is that of a canonical isomorphism $\Psi_{\sigma_1, \sigma_2}: \sigma_2^\boxtimes \xrightarrow{\sim} \sigma_1^\boxtimes$ associated to any pair of labeled sequences σ_1, σ_2 such that $|\sigma_1| = |\sigma_2|$. This construction is carried out in §4. For the rest of this section we concentrate on developing the tools used in its construction.

3.2. Compositions and fibered products of labeled maps. Here we give a collection of basic notions involving labeled maps and sequences. Mainly, we extend the isomorphisms of the input data in §2.1 and their compatibilities to the *labeled* setup. Thus the isomorphisms defined here are not really new, but are simply repackaged versions of the old ones designed to work in the new setup.

3.2.1. Let $\sigma_1 = F_1, \dots, F_n$ and $\sigma_2 = G_1, \dots, G_m$ be labeled sequences such that the target of σ_1 equals the source of σ_2 . We may then concatenate the two sequences in the obvious manner resulting in a labeled sequence

$$\sigma_1 \star \sigma_2 := F_1, \dots, F_n, G_1, \dots, G_m.$$

Note that $|\sigma_1 \star \sigma_2| = |\sigma_2| |\sigma_1|$ and $(\sigma_1 \star \sigma_2)^\boxtimes = \sigma_2^\boxtimes \sigma_1^\boxtimes$.

3.2.2. For *equi*-labeled maps $X \xrightarrow{(f_1, \lambda)} Y \xrightarrow{(f_2, \lambda)} Z$, with $f_3 := f_2 f_1$ and $F_i := (f_i, \lambda)$ we define a comparison isomorphism $C_{F_1, F_2}: F_1^\boxtimes F_2^\boxtimes \xrightarrow{\sim} F_3^\boxtimes$ by

$$C_{F_1, F_2} = \begin{cases} C_{f_1, f_2}^\times & \text{if } \lambda_i = \text{P}; \\ C_{f_1, f_2}^\square & \text{if } \lambda_i = \text{O}. \end{cases}$$

3.2.3. Let $X \xrightarrow{(f_1, \lambda_1)} Y \xrightarrow{(f_2, \lambda_2)} X$ be labeled maps such that $\lambda_1 = \text{P}$ and $f_2 f_1 = 1_X$. Set $F_i := (f_i, \lambda_i)$. Then we define a fundamental isomorphism

$$(3.2.3.1) \quad \phi_{F_1, F_2}: F_1^\boxtimes F_2^\boxtimes \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$$

by

$$\phi_{F_1, F_2} = \begin{cases} C_{f_1, f_2}^\times & \text{if } \lambda_2 = \text{P}; \\ \phi_{f_1, f_2} & \text{if } \lambda_2 = \text{O}. \end{cases}$$

3.2.4. A *labeled cartesian square* is a quadruplet $\mathfrak{s} = (F_1, F_2, F'_1, F'_2)$ of labeled maps with

$$F_1 = (f_1, \lambda_1), \quad F_2 = (f_2, \lambda_2), \quad F'_1 = (f'_1, \lambda'_1), \quad F'_2 = (f'_2, \lambda'_2),$$

that fit into a diagram as follows

$$(3.2.4.1) \quad \begin{array}{ccc} U & \xrightarrow{F'_2} & X \\ F'_1 \downarrow & & \downarrow F_1 \\ V & \xrightarrow{F_2} & Y \end{array}$$

such that the underlying diagram of \mathbf{C} -maps is a usual cartesian square and $\lambda'_i = \lambda_i$. We shall use the same name (such as \mathfrak{s}) to denote a labeled cartesian square and the underlying square of unlabeled maps. We also say that \mathfrak{s} is a labeled cartesian square on (F_1, F_2) . By default, the first and third element of the quadruplet underlying \mathfrak{s} are drawn vertically and the other two horizontally. Via this convention, the diagram uniquely determines the quadruplet.

The *transpose* of a labeled cartesian square $\mathfrak{s} = (F_1, F_2, F'_1, F'_2)$ is the labeled cartesian square $\mathfrak{s}^t := (F_2, F_1, F'_2, F'_1)$.

Any two labeled maps F_1, F_2 having the same target, give rise to a labeled cartesian square on (F_1, F_2) , in the obvious way. Of course, the new ingredients of the square are not uniquely determined though they are unique up to a unique isomorphism.

From now on, we shall implicitly assume all maps and cartesian squares to be labeled ones.

For the cartesian square \mathfrak{s} in (3.2.4.1), we define a base-change isomorphism

$$(3.2.4.2) \quad \beta_{\mathfrak{s}} : F'_2 \boxtimes F'_1 \xrightarrow{\sim} F'_1 \boxtimes F'_2$$

by

$$\beta_{\mathfrak{s}} = \begin{cases} (C_{f'_1, f'_2}^\times)^{-1} C_{f'_2, f'_1}^\times, & \text{if } \lambda_1 = \lambda_2 = \mathbf{P}; \\ (C_{f'_1, f'_2}^\square)^{-1} C_{f'_2, f'_1}^\square, & \text{if } \lambda_1 = \lambda_2 = \mathbf{O}; \\ \beta_{\mathfrak{s}}, & \text{if } \lambda_1 = \mathbf{P}, \lambda_2 = \mathbf{O}; \\ \beta_{\mathfrak{s}^t}^{-1}, & \text{if } \lambda_1 = \mathbf{O}, \lambda_2 = \mathbf{P}. \end{cases}$$

Note that β_{-} is symmetric, i.e., $\beta_{\mathfrak{s}} = \beta_{\mathfrak{s}^t}^{-1}$.

The isomorphisms $C_{-, -}$, $\phi_{-, -}$ and β_{-} defined above satisfy many compatibilities including those analogous to the ones in §2.1. We record some of these.

LEMMA 3.2.5.

(i) *Let \mathfrak{s} be a labeled cartesian square as follows*

$$\begin{array}{ccc} X' & \xrightarrow{F} & X \\ I' \downarrow & & \downarrow I \\ X' & \xrightarrow{F} & X \end{array}$$

where I, I' are identity maps with label \mathbf{P} . Then the natural isomorphism

$$F^\boxtimes = F^\boxtimes I^\boxtimes \xrightarrow{\beta_{\mathfrak{s}}} I'^\boxtimes F^\boxtimes = F^\boxtimes$$

is the identity.

- (ii) The fundamental isomorphism ϕ_{F_1, F_2} of (3.2.3.1) is compatible with base change on X by a labeled map, i.e., for any diagram of labeled cartesian squares as follows, where the top row composes to $1_{X'}$, \mathfrak{s}_1 is the square on the left and \mathfrak{s}_2 the one on the right,

$$\begin{array}{ccccc} X' & \xrightarrow{F'_1} & Y' & \xrightarrow{F'_2} & X' \\ F_3 \downarrow & & F'_3 \downarrow & & F_3 \downarrow \\ X & \xrightarrow{F_1} & Y & \xrightarrow{F_2} & X \end{array}$$

the following diagram of isomorphisms commutes.

$$\begin{array}{ccccc} F'_1 \boxtimes F'_2 \boxtimes F'_3 & \xrightarrow{F'_1 \boxtimes (\beta_{\mathfrak{s}_2})} & F'_1 \boxtimes F'_3 \boxtimes F'_2 & \xrightarrow{\beta_{\mathfrak{s}_1}(F'_2 \boxtimes)} & F'_3 \boxtimes F'_1 \boxtimes F'_2 \\ \downarrow \phi_{F'_1, F'_2}(F'_3 \boxtimes) & & & & \downarrow F'_3 \boxtimes (\phi_{F_1, F_2}) \\ \mathbf{1}_{\mathcal{D}_X}, F'_3 & \xlongequal{\quad} & F'_3 & \xlongequal{\quad} & F'_3 \mathbf{1}_{\mathcal{D}_X} \end{array}$$

- (iii) The base-change isomorphism of (3.2.4.2) is horizontally and vertically transitive. For instance, consider a horizontal extension to the diagram \mathfrak{s} in (3.2.4.1) as follows where F_3 has the same label λ_2 as F_2 and the appended square, is also cartesian.

$$\begin{array}{ccccc} U_1 & \xrightarrow{F'_3} & U & \xrightarrow{F'_2} & X \\ F'_1 \downarrow & & F'_1 \downarrow & & \downarrow F_1 \\ V_1 & \xrightarrow{F_3} & V & \xrightarrow{F_2} & Y \end{array}$$

Let F_4, F'_4 be the composite maps $|F_2||F_3|, |F'_2||F'_3|$ respectively with label λ_2 . Let \mathfrak{s}_1 be the appended square on the left and \mathfrak{c} the composite square. Then the following diagram commutes, where

$$\begin{array}{ccccc} F'_3 \boxtimes F'_2 \boxtimes F'_1 & \xrightarrow{F'_3 \boxtimes (\beta_{\mathfrak{s}})} & F'_3 \boxtimes F'_1 \boxtimes F'_2 & \xrightarrow{\beta_{\mathfrak{s}_1}(F'_2 \boxtimes)} & F'_1 \boxtimes F'_3 \boxtimes F'_2 \\ \downarrow C_{F'_3, F'_2}(F'_1 \boxtimes) & & & & \downarrow F'_1 \boxtimes (C_{F_3, F_2}) \\ F'_4 \boxtimes F'_1 & \xrightarrow{\beta_{\mathfrak{c}}} & & & F'_1 \boxtimes F'_4 \end{array}$$

Similarly, transitivity holds for vertical extensions of \mathfrak{s} .

PROOF. (i). Let $F = (f, \lambda)$. If $\lambda = \mathbf{P}$ then we conclude using pseudofunctoriality of $(-)^{\times}$. Assume $\lambda = \mathbf{O}$. Let b denote the automorphism of F^{\boxtimes} induced by $\beta_{\mathfrak{s}}$. From the vertical transitivity property of β_{-} (§2.1, [C](ii)) corresponding to the following diagram,

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \parallel & & \parallel \\ X' & \xrightarrow{f} & X \\ \parallel & & \parallel \\ X' & \xrightarrow{f} & X \end{array}$$

we conclude that $bb = b$. Since b is an isomorphism, therefore it is the identity.

(ii). Recall that $\lambda_1 = P$ by hypothesis in (3.2.3.1). If $\lambda_2 = O$, then the asserted compatibility follows from parts (a) or (b) of §2.1[D](i), depending on whether λ_3 equals P or O . Now assume $\lambda_2 = P$. If $\lambda_3 = O$, then we conclude by the vertical transitivity property of §2.1[C](i) and by part (i) above. If $\lambda_3 = P$, then all the labels are P and we conclude by pseudofunctoriality.

(iii). We argue as in (ii) by looking at various cases depending on the value of the labels involved. \square

3.3. Fibered products of sequences. We consider diagrams resulting from taking fibered products of two sequences having the same target and associate a generalized base-change isomorphism to such diagrams.

3.3.1. Let σ_1, σ_2 be two labeled sequences having the same target T . Then one associates a cartesian diagram to these two sequences as follows. Suppose σ_1, σ_2 are given, respectively, by

$$\begin{aligned} Y_1 &\xrightarrow{(f_1, \alpha_1)} Y_2 \xrightarrow{(f_2, \alpha_2)} Y_3 \longrightarrow \cdots \longrightarrow Y_n \xrightarrow{(f_n, \alpha_n)} Y_{n+1} = T, \\ Z_1 &\xrightarrow{(g_1, \gamma_1)} Z_2 \xrightarrow{(g_2, \gamma_2)} Z_3 \longrightarrow \cdots \longrightarrow Z_m \xrightarrow{(g_m, \gamma_m)} Z_{m+1} = T. \end{aligned}$$

Then a *fiber-product diagram on σ_1, σ_2* (or *cartesian diagram on σ_1, σ_2*) is an $n \times m$ array $\mathfrak{S} = [\mathfrak{s}_{i,j}]$ of labeled cartesian squares as follows where $\mathfrak{s}_{i,j}$ is the square that is the i -th one from the top and the j -th one from the left,

$$\begin{array}{ccccccc} X_{1,1} & \xrightarrow{H_{1,1}} & X_{1,2} & \xrightarrow{H_{1,2}} & \cdots & \xrightarrow{H_{1,m-1}} & X_{1,m} & \xrightarrow{H_{1,m}} & X_{1,m+1} \\ \downarrow V_{1,1} & & \downarrow V_{1,2} & & & & \downarrow V_{1,m} & & \downarrow V_{1,m+1} \\ X_{2,1} & \xrightarrow{H_{2,1}} & X_{2,2} & \xrightarrow{H_{2,2}} & \cdots & \xrightarrow{H_{2,m-1}} & X_{2,m} & \xrightarrow{H_{2,m}} & X_{2,m+1} \\ \downarrow V_{2,1} & & \downarrow V_{2,2} & & & & \downarrow & & \downarrow V_{2,m+1} \\ X_{3,1} & \longrightarrow & X_{3,2} & \longrightarrow & \cdots & \longrightarrow & X_{3,m} & \longrightarrow & X_{3,m+1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \cdots & & \cdots & & & & \cdots & & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X_{n,1} & \longrightarrow & X_{n,2} & \longrightarrow & \cdots & \longrightarrow & X_{n,m} & \longrightarrow & X_{n,m+1} \\ \downarrow V_{n,1} & & \downarrow V_{n,2} & & & & \downarrow V_{n,m} & & \downarrow V_{n,m+1} \\ X_{n+1,1} & \xrightarrow{H_{n+1,1}} & X_{n+1,2} & \xrightarrow{H_{n+1,2}} & \cdots & \xrightarrow{H_{n+1,m-1}} & X_{n+1,m} & \xrightarrow{H_{n+1,m}} & X_{n+1,m+1} \end{array}$$

and such that the following two conditions are satisfied.

- The rightmost column of maps is σ_1 ,
i.e., $X_{i,m+1} = Y_i$ and $V_{i,m+1} = (f_i, \alpha_i)$.
- The bottommost row of maps is σ_2 ,
i.e., $X_{n+1,j} = Z_j$ and $H_{n+1,j} = (g_j, \gamma_j)$.

It follows from the definitions that $X_{i,j} = Y_i \times_T Z_j$ and that for all i, j , $V_{i,j}$ has label α_i and $H_{i,j}$ has label γ_j . We denote the underlying maps of $V_{i,j}$ and $H_{i,j}$

by $v_{i,j}$ and $h_{i,j}$ respectively. We denote the i -th row of horizontal maps by H_i (i.e., H_i is the sequence consisting of $H_{i,j}$) and the j -th column of vertical maps by V_j .

The existence of a fiber-product diagram associated to σ_1, σ_2 is obvious. As in the case of products of single maps, the fiber-product diagram \mathcal{S} is not unique as a function of σ_1 and σ_2 though it is unique up to a unique isomorphism (at each vertex). In particular, the terms such as $V_{i,j}, H_{i,j}$ are really functions of \mathcal{S} .

Sometimes we use draw \mathcal{S} in a more compact form as follows:

$$\begin{array}{ccc} X_{1,1} & \xRightarrow{\sigma_4} & Y_1 \\ \sigma_3 \downarrow & & \downarrow \sigma_1 \\ Z_1 & \xRightarrow{\sigma_2} & T \end{array}$$

where $\sigma_3 := V_1(\mathcal{S})$ and $\sigma_4 := H_1(\mathcal{S})$. This diagram suppresses a lot of information on \mathcal{S} but is useful when the extra information is not directly needed. By abuse of language, we also call \mathcal{S} a cartesian square (of double arrows) on (σ_1, σ_2) . The usual cartesian square on single arrows is then called a unit square.

3.3.2. Our next goal is to associate to the above diagram \mathcal{S} , a generalized base-change isomorphism

$$(3.3.2.1) \quad \beta_{\mathcal{S}} : H_1^{\boxtimes} V_{m+1}^{\boxtimes} \xrightarrow{\sim} V_1^{\boxtimes} H_{n+1}^{\boxtimes}.$$

The idea behind defining $\beta_{\mathcal{S}}$ is quite simple. Let us think of \mathcal{S} as a directed planar graph in the obvious sense. Then the sequences $H_1 \star V_{m+1}$ and $V_1 \star H_{n+1}$ are the two outermost directed paths from $X_{1,1}$ to $X_{n+1,m+1}$. Now the base-change isomorphism $\beta_{\mathfrak{s}_{i,j}}$ associated to $\mathfrak{s}_{i,j}$ may be thought of as a “flip” from $H_{i,j} \star V_{i,j+1}$ to $V_{i,j} \star H_{i+1,j}$ (as directed paths between $X_{i,j}$ and $X_{i+1,j+1}$). Defining $\beta_{\mathcal{S}}$ now simply amounts to choosing a sequence of flips that transforms $H_1 \star V_{m+1}$ to $V_1 \star H_{n+1}$. Any such sequence of flips/squares necessarily starts from $\mathfrak{s}_{1,m}$ and ends with $\mathfrak{s}_{n,1}$ and uses each of the mn squares exactly once. But the order in which the intermediate flips can be chosen is not unique.

Thus, what needs to be addressed is why the resulting definition of $\beta_{\mathcal{S}}$ is independent of the choice of the sequence of flips. As it turns out this holds for simple functorial reasons, i.e., no special property of the base-change isomorphisms for unit squares, other than functoriality, comes into play. Indeed, the question essentially reduces to an elementary graph-theoretic exercise. Since we use this kind of argument on a separate occasion too, we provide a sketch of the details involved.

3.3.3. This is an interlude on some graph-theoretic notions needed for resolving the question of well-definedness of $\beta_{\mathcal{S}}$. We use the same notation as above. For convenience, we work more generally with directed paths between two arbitrary vertices in \mathcal{S} . Containment of paths is defined in the obvious manner. Now we discuss some terminology.

- Given two (directed) paths p, p' , we say $p' \geq p$ if both share the same starting vertex and the same ending vertex and if for any vertex $X_{i,j}$ in p there exists a vertex $X_{i',j'}$ in p' such that $i' \leq i$ and $j' \geq j$. Clearly \geq is reflexive and transitive. Moreover, if $p' \geq p$ then p' and p do not cross each other anywhere. In fact, for any vertex common to p and p' , say $X_{i,j}$, the vertical edge $V_{i-1,j}$ ending at $X_{i,j}$ is in p only if it is also in p' and the horizontal edge $H_{i,j}$ starting from $X_{i,j}$ is in p only if it is also in p' . One easily concludes that \geq is in fact antisymmetric, i.e., if $p \geq p'$ and $p' \geq p$ then $p = p'$.

Thus \geq is a partial ordering on paths. We say $p' > p$ if $p' \geq p$ and $p' \neq p$.

- We say that a square $\mathfrak{s}_{i,j}$ flips a path p_1 to another one p_2 or that $\mathfrak{s}_{i,j}$ flops p_2 to p_1 if the north-eastern half of the boundary of $\mathfrak{s}_{i,j}$, viz. $H_{i,j} \star V_{i,j+1}$, is contained in p_1 , the other half $V_{i,j} \star H_{i+1,j}$ is in p_2 and if p_2 is obtained from p_1 by interchanging these halves.

- A square $\mathfrak{s} = \mathfrak{s}_{i,j}$ is said to be *open* for a path p if $H_{i,j} \star V_{i,j+1}$ is contained in p . This is equivalent to saying that there exists another path p' such that \mathfrak{s} flips p to p' . Note that, in this situation, if \mathfrak{t} is another square distinct from \mathfrak{s} that is also open for p , then \mathfrak{t} is also open for p' .

- A path p_1 is said to be *adjacent* to another path p_2 if $p_1 > p_2$ and for any path p such that $p_1 \geq p \geq p_2$ we have $p = p_1$ or $p = p_2$. It is easily verified that p_1 is adjacent to p_2 if and only if there exists a square \mathfrak{s} , necessarily unique, such that \mathfrak{s} flips p_1 to p_2 .

- A *maximal* chain of paths is a sequence $p_1 > p_2 > \dots > p_N$, ($N \geq 1$) such that p_i is adjacent to p_{i+1} for all i .

- Two maximal chains

$$c := p_1 > p_2 > \dots > p_N \quad \text{and} \quad c' := p'_1 > p'_2 > \dots > p'_N$$

are said to be *flip-flop neighbors* if one of the following conditions holds:

- (a) $c = c'$; or
- (b) $N > 2$ and there exists an integer i such that $1 < i < N$ and such that $p_j = p'_j$ for $j \neq i$.

In the situation of (b), with $c \neq c'$, if \mathfrak{t} (resp. \mathfrak{s}) is the square that flips p_{i-1} to p_i (resp. p_i to p_{i+1}), then \mathfrak{s} (resp. \mathfrak{t}) flips p'_{i-1} to p'_i (resp. p'_i to p'_{i+1}). Moreover, \mathfrak{t} and \mathfrak{s} do not share a common edge.

Here then is the main graph-theoretic result that we use.

LEMMA 3.3.4. *For any two maximal chains*

$$c := p_1 > p_2 > \dots > p_N \quad \text{and} \quad c' := p'_1 > p'_2 > \dots > p'_N$$

satisfying $p_1 = p'_1$, $p_N = p'_N$ there exists a sequence of maximal chains c_1, c_2, \dots, c_M such that $c_1 = c$, $c_M = c'$ and such that for all i , c_i and c_{i+1} are flip-flop neighbors.

PROOF. We use induction on N , the length of the chains c, c' . If $N \leq 3$ then the lemma holds trivially. Assume $N > 3$. We claim that there exists another maximal chain $c'' := p''_1 > p''_2 > \dots > p''_N$ satisfying $p''_1 = p_1, p''_N = p_N$ and there are integers i, j such that $1 < i, j < N$ and such that $p''_i = p_i$ and $p''_j = p'_j$.

Assuming the above claim we proceed as follows. It suffices to find a sequence of flip-flop neighbors between c, c'' and between c'', c' . In the case of c and c'' we may break up each of these chains into two pieces as follows

$$\begin{aligned} c_{\leq i} &= p_1 > \dots > p_i, & c_{\geq i} &= p_i > \dots > p_N, \\ c''_{\leq i} &= p''_1 > \dots > p''_i, & c''_{\geq i} &= p''_i > \dots > p''_N. \end{aligned}$$

By induction hypothesis there exists a sequence of flip-flop neighbors between $c_{\leq i}$ and $c''_{\leq i}$ and also one between $c_{\geq i}$ and $c''_{\geq i}$. These can be put together to give one for c, c'' . Similarly one argues for c'', c' .

It remains to prove the above claim. If $p'_2 = p_2$ then the claim is proven by choosing $c'' = c'$, and $i = j = 2$. Assume $p'_2 \neq p_2$. Let $\mathfrak{s}_{(k)}$ (resp. \mathfrak{s}') be the square that flips p_k to p_{k+1} (resp. p'_1 to p'_2). Then \mathfrak{s}' is open for p_1 but not for $p_N = p'_N$. Let i be the largest integer such that \mathfrak{s}' is open for p_i . Then $\mathfrak{s}' = \mathfrak{s}_{(i)}$. We

now define c'' inductively as follows. By definition, $p_k'' = p_k$ for $k = 1, N$. Let p_2'' be the path obtained by flipping p_1 via \mathfrak{s}' . For $1 < k < i + 1$, let p_{k+1}'' be the path obtained by flipping p_k'' via $\mathfrak{s}_{(k-1)}$ and for $k \geq i + 1$, let p_{k+1}'' be obtained by flipping p_k'' via $\mathfrak{s}_{(k)}$. Then for this choice of c'' and i and with $j = 2$, the claim is verified. \square

3.3.5. We return to the issue of the well-definedness of $\beta_{\mathfrak{s}}$. For any path p from $X_{1,1}$ to $X_{n+1,m+1}$, let p^{\boxtimes} be the obvious functor $\mathcal{D}_{X_{n+1,m+1}} \rightarrow \mathcal{D}_{X_{1,1}}$ induced via p . Then for any maximal chain c given by say, $p_1 > p_2 > \dots > p_{n+m}$, where $p_1 = H_1 \star V_{m+1}$ and $p_{n+m} = V_1 \star H_{n+1}$, if $\mathfrak{s}_{(k)}$ is the square that flips p_k to p_{k+1} then one obtains an isomorphism $\beta_{\mathfrak{s}}(c)$ given by

$$p_1^{\boxtimes} \xrightarrow{\text{via } \beta_{\mathfrak{s}_{(1)}}} p_2^{\boxtimes} \xrightarrow{\text{via } \beta_{\mathfrak{s}_{(2)}}} \dots \xrightarrow{\text{via } \beta_{\mathfrak{s}_{(n+m-1)}}} p_{n+m}^{\boxtimes}.$$

Thus the problem may be rephrased as saying that for any two maximal chains c, c' that start from $H_1 \star V_{m+1}$ and end with $V_1 \star H_{n+1}$, we have $\beta_{\mathfrak{s}}(c) = \beta_{\mathfrak{s}}(c')$.

By Lemma 3.3.4, it suffices to prove equality in the case where c and c' are flip-flop neighbors and $c \neq c'$. Let p_k (resp. p'_k) be the k -th element of the chain c (resp. c'). Let i be the unique integer such that $p_i \neq p'_i$. If \mathfrak{r} is the square that flips p_{i-1} to p_i and \mathfrak{s} the one that flips p_i to p_{i+1} , then the following diagram commutes for functorial reasons.

$$\begin{array}{ccc} p_{i-1}^{\boxtimes} & \xrightarrow{\text{via } \beta_{\mathfrak{r}}} & p_i^{\boxtimes} \\ \text{via } \beta_{\mathfrak{s}} \downarrow & & \downarrow \text{via } \beta_{\mathfrak{s}} \\ p_i'^{\boxtimes} & \xrightarrow{\text{via } \beta_{\mathfrak{r}}} & p_{i+1}^{\boxtimes} \end{array}$$

Now $\beta_{\mathfrak{s}}(c) = \beta_{\mathfrak{s}}(c')$ follows immediately.

The following lemma generalizes 3.2.5(i) and is used in §5.

LEMMA 3.3.6. *Let $\sigma: Z \rightrightarrows X$ be a labeled sequence. Corresponding to any cartesian diagram \mathfrak{S} as follows where the vertical maps in the intermediate places are all P-labeled identity maps,*

$$\begin{array}{ccc} Z & \xrightleftharpoons{\sigma} & X \\ I_1 = (1_Z, P) \parallel & & \parallel I_2 = (1_X, P) \\ Z & \xrightleftharpoons{\sigma} & X \end{array}$$

the isomorphism $\sigma^{\boxtimes} = \sigma^{\boxtimes} I_2^{\boxtimes} \xrightarrow{\beta_{\mathfrak{S}}} I_1^{\boxtimes} \sigma^{\boxtimes} = \sigma^{\boxtimes}$ is the identity.

PROOF. We prove by induction on the length of σ . The case of length 1 has been dealt with in 3.2.5(i). Now suppose $\sigma = \sigma_1 \star \sigma_2$. Consider the following associated cartesian diagram with obvious notation. Let \mathfrak{S}_1 denote the square on the left and \mathfrak{S}_2 the one on the right.

$$\begin{array}{ccccc} Z & \xrightarrow{\sigma_1} & Y & \xrightarrow{\sigma_2} & X \\ I_1 \parallel & & J \parallel & & \parallel I_2 \\ Z & \xrightarrow{\sigma_1} & Y & \xrightarrow{\sigma_2} & X \end{array}$$

The assertion of the lemma amounts to checking that the outer border of the following diagram commutes.

$$\begin{array}{ccccc}
\sigma_1^\boxtimes \sigma_2^\boxtimes & \xlongequal{\quad} & \sigma_1^\boxtimes \sigma_2^\boxtimes & \xlongequal{\quad} & \sigma_1^\boxtimes \sigma_2^\boxtimes \\
\parallel & \downarrow \dagger_2 & \parallel & \downarrow \dagger_1 & \parallel \\
\sigma_1^\boxtimes \sigma_2^\boxtimes I_2^\boxtimes & \xrightarrow[\sigma_1^\boxtimes(\beta_{s_2})]{} & \sigma_1^\boxtimes J^\boxtimes \sigma_2^\boxtimes & \xrightarrow[\beta_{s_1}(\sigma_2^\boxtimes)]{} & I_1^\boxtimes \sigma_1^\boxtimes \sigma_2^\boxtimes
\end{array}$$

Since σ_i has length smaller than σ , by induction hypothesis the conclusion holds for σ_i and hence \dagger_i commutes. \square

3.4. The generalized fundamental isomorphism. We upgrade the fundamental isomorphism of (3.2.3.1) to the level of sequences of arbitrary length. Modulo the issue of canonicity, the generalization is achieved in (3.4.1.5). The isomorphism defined here is one of the important special cases as well as a building block of our central construction of $\Psi_{-, -}$ in §4.1.

3.4.1. Consider a labeled sequence σ given by

$$(3.4.1.1) \quad X = Y_1 \xrightarrow{(f_1, \lambda_1)} Y_2 \xrightarrow{(f_2, \lambda_2)} Y_3 \longrightarrow \cdots \longrightarrow Y_n \xrightarrow{(f_n, \lambda_n)} Y_{n+1} = X$$

such that $|\sigma| := 1_X$. Our aim is to define an isomorphism $\sigma^\boxtimes \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$ that generalizes the fundamental isomorphism of (3.2.3.1). This is accomplished by means of a staircase diagram that we now define.

A *staircase diagram based on σ* is a collection of $n(n-1)/2$ cartesian squares stacked into n columns as follows, (where the i -th column from the left has $n-i$ squares in it)

$$\begin{array}{ccccccc}
(3.4.1.2) & X & & & & & \\
& \downarrow V_{1,1} & & & & & \\
& Y_{2,1} & \xrightarrow{H_{2,1}} & X & & & \\
& \downarrow V_{2,1} & & \downarrow V_{2,2} & & & \\
& \cdots & & \cdots & & \cdots & \\
& \downarrow & & \downarrow & & & \\
& Y_{n-1,1} & \longrightarrow & Y_{n-1,2} & \longrightarrow & \cdots & \longrightarrow X \\
& \downarrow V_{n-1,1} & & \downarrow V_{n-1,2} & & & \downarrow V_{n-1,n-1} \\
& Y_{n,1} & \xrightarrow{H_{n,1}} & Y_{n,2} & \xrightarrow{H_{n,2}} & \cdots & \xrightarrow{H_{n,n-2}} Y_{n,n-1} \xrightarrow{H_{n,n-1}} X \\
& \downarrow V_{n,1} & & \downarrow V_{n,2} & & & \downarrow V_{n,n-1} & \downarrow V_{n,n} \\
& X & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow Y_{n-1} & \longrightarrow Y_n & \longrightarrow X
\end{array}$$

satisfying the following conditions.

- The sequence of maps at the base is σ . Thus we may set $H_{n+1,i} := (f_i, \lambda_i)$.
- The vertical maps, $V_{i,j}$ all have label P.
- Let $v_{i,j} = |V_{i,j}|$ and $h_{i,j} = |H_{i,j}|$. Then for every i , the following hold.

$$h_{i+1,i} v_{i,i} = 1_X \quad v_{n,i} v_{n-1,i} \cdots v_{i,i} = f_{i-1} f_{i-2} \cdots f_1$$

A quick approach to constructing a staircase is obtained by noting that $Y_{i,j}$ must equal the product $Y_j \times_{Y_i} X$ where $Y_j \rightarrow Y_i$ (resp. $X \rightarrow Y_i$) is the composite map $f_i \cdots f_j$ (resp. $f_i \cdots f_1$); then $h_{i,j}$ and $v_{i,j}$ are the obvious natural maps induced by f_j and f_i respectively. It also follows that for a fixed σ , any two staircases based on it are isomorphic via unique isomorphisms at each vertex.

Nevertheless, the lack of uniqueness of a staircase, means that all the terms used in definition of a staircase \mathcal{S} such as $V_{-,-}$, $H_{-,-}$, $Y_{-,-}$, etc., should be regarded as functions of \mathcal{S} , e.g., $V_{-,-}(\mathcal{S})$. In specific contexts, when the choice of \mathcal{S} is known, reference to \mathcal{S} in the notation may be dropped.

For σ of (3.4.1.1), let us assume that a staircase \mathcal{S} such as the one drawn above has been chosen. We now show how \mathcal{S} gives rise to an isomorphism $\sigma^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$.

Each “step” in \mathcal{S} gives rise to a functor isomorphic to $\mathbf{1}_{\mathcal{D}_X}$, i.e., for $1 \leq i \leq n$, we have $h_{i+1,i} v_{i,i} = 1_X$ and hence (3.2.3.1) gives an isomorphism

$$V_{i,i}^{\boxtimes} H_{i+1,i}^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}.$$

Set $\mathbf{Steps} = (V_{1,1} \star H_{2,1}) \star (V_{2,2} \star H_{3,2}) \star \cdots \star (V_{n,n} \star H_{n+1,n})$ and let

$$(3.4.1.3) \quad \mathbf{Steps}^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$$

be the isomorphism obtained by successively using (3.2.3.1).

Let V_1 be the sequence occurring as the leftmost column of maps in \mathcal{S} . By definition, V_1 composes to 1_X and the label of each map in it is \mathbf{P} . Therefore pseudofunctoriality of $(-)^{\times}$ gives a natural isomorphism

$$(3.4.1.4) \quad V_1^{\boxtimes} \xrightarrow{\sim} (1_X)^{\times} = \mathbf{1}_{\mathcal{D}_X}.$$

The *generalized* fundamental isomorphism associated to a staircase \mathcal{S} based on σ is defined to be the isomorphism

$$(3.4.1.5) \quad \Phi_{\sigma}(\mathcal{S}): \sigma^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$$

obtained as the following composition

$$\sigma^{\boxtimes} \xrightarrow{\text{via (3.4.1.4)}^{-1}} V_1^{\boxtimes} \sigma^{\boxtimes} \longrightarrow \mathbf{Steps}^{\boxtimes} \xrightarrow{(3.4.1.3)} \mathbf{1}_{\mathcal{D}_X}$$

where the map in the middle is defined the same way as $\beta_{\mathcal{S}}$ is defined in (3.3.2.1), viz., through the base-change isomorphisms associated to each of the unit squares in \mathcal{S} . (As in the case of $\beta_{\mathcal{S}}$, the order in which the squares are chosen can vary but the end result is the same in view of the arguments in 3.3.5.)

A-priori, for a fixed σ , the generalized fundamental isomorphism depends on the choice of the staircase \mathcal{S} based on σ , in that, if \mathcal{S}' is another staircase based on σ , then it is not clear whether $\Phi_{\sigma}(\mathcal{S}) = \Phi_{\sigma}(\mathcal{S}')$. We would like to say that the two are indeed equal so that (3.4.1.5) defines a canonical isomorphism, independent of the choice of \mathcal{S} . Fortunately, this is true, though it is not altogether trivial to verify. We take up this issue in the next subsection.

Let us verify that $\Phi_{\sigma}(\mathcal{S})$ does indeed generalize the earlier definitions of a fundamental isomorphism.

LEMMA 3.4.2. *Let σ be the sequence $X \xrightarrow{F_1} Y \xrightarrow{F_2} X$ of 3.2.3, i.e., F_1 has label \mathbf{P} and $|F_1 \star F_2| = 1_X$. Let \mathcal{S} be a staircase based on σ . Then $\Phi_{\sigma}(\mathcal{S}) = \phi_{F_1, F_2}$.*

PROOF. Suppose \mathcal{S} is given as follows. Let \mathfrak{s} be the cartesian square in it.

$$\begin{array}{ccccc} & X & & & \\ & \Delta \downarrow & & & \\ & X^2 & \xrightarrow{Q} & X & \\ P \downarrow & & & & \downarrow F'_1 = F_1 \\ X & \xrightarrow{F_1} & Y & \xrightarrow{F_2} & X \end{array}$$

By construction, Δ , P , Q and F_1 , all have label \mathbf{P} . Hence the following composite isomorphism θ ,

$$F_1^\boxtimes \xrightarrow{\text{via } C_{\Delta, P}^{-1}} \Delta^\boxtimes P^\boxtimes F_1^\boxtimes \xrightarrow{\text{via } \beta_\mathfrak{s}} \Delta^\boxtimes Q^\boxtimes F_1'^\boxtimes \xrightarrow{\text{via } \phi_{\Delta, Q}} F_1'^\boxtimes,$$

is the identity for pseudofunctorial reasons. By definition, $\Phi_\sigma(\mathcal{S})$ is the following composition and thus the lemma follows.

$$F_1^\boxtimes F_2^\boxtimes \xrightarrow{\text{via } \theta \text{ above}} F_1'^\boxtimes F_2^\boxtimes \xrightarrow{\phi_{F_1', F_2}} \mathbf{1}_{\mathcal{D}_X}$$

□

3.5. Canonicity of the fundamental isomorphism. Our main concern here is of showing that the generalized fundamental isomorphism of (3.4.1.5) is independent of the choice of the staircase diagram chosen. We accomplish this in Proposition 3.5.9 below.

The general idea is of comparing the base-change isomorphism β_- resulting from isomorphic cartesian squares. We begin by setting up a comparison isomorphism $C_{-, -}$ for the composition of two maps where one of them is an isomorphism with label \mathbf{P} .

In what follows, in the context of labeled maps, a vertically drawn “equal sign” is downward pointing by default and a horizontally drawn “equal sign” points rightward.

DEFINITION 3.5.1. Let $i: Y \rightarrow Y'$ and $j: Y' \rightarrow Y$ be inverse isomorphisms. Set $I := (i, \mathbf{P})$, $J := (j, \mathbf{P})$.

- (i) For any labeled map $X \xrightarrow{F=(f, \lambda)} Y$, with F' as the map $X \xrightarrow{if} Y'$ with label λ , (since i is in both, \mathbf{P} and \mathbf{O} , hence F' can have the same label as F) we define $C_{F, I}: F^\boxtimes I^\boxtimes \xrightarrow{\sim} F'^\boxtimes$ to be the base-change isomorphism $\beta_\mathfrak{s}$ associated to the cartesian square \mathfrak{s} drawn as follows.

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ (1_X, \mathbf{P}) \parallel & & \downarrow I \\ X & \xrightarrow{F'} & Y' \end{array}$$

- (ii) For any labeled map $Y \xrightarrow{G=(g, \lambda)} Z$, with $G' := (gj, \lambda): Y' \rightarrow Z$, we define $C_{J, G}: J^\boxtimes G^\boxtimes \xrightarrow{\sim} G'^\boxtimes$ to be the base-change isomorphism $\beta_\mathfrak{s}$ associated

to the cartesian square \mathfrak{s} drawn as follows.

$$\begin{array}{ccc} Y' & \xrightarrow{J} & Y \\ G' \downarrow & & \downarrow G \\ Z & \xrightarrow{(1_Z, P)} & Z \end{array}$$

REMARKS 3.5.2.

1. In 3.5.1(i), by interchanging the role of i with j and F with F' we obtain in an analogous fashion an isomorphism $C_{F',J}: F'^{\boxtimes} J^{\boxtimes} \xrightarrow{\sim} F^{\boxtimes}$. Likewise, in (ii) we obtain an isomorphism $C_{I,G'}: I^{\boxtimes} G'^{\boxtimes} \xrightarrow{\sim} G^{\boxtimes}$.

2. In 3.5.1(i), if F has label P , then every map in \mathfrak{s} has label P and hence by pseudofunctoriality of $(-)^{\boxtimes}$ we see that the above definition of $C_{F,I}$ agrees with the one in 3.2.2. A similar remark holds for $C_{G,J}$ of 3.5.1(ii).

LEMMA 3.5.3. *Let notation be as in 3.5.1. Let the following isomorphisms*

$$\varphi: I^{\boxtimes} J^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_Y} \quad \text{and} \quad \psi: J^{\boxtimes} I^{\boxtimes} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_{Y'}}$$

be the ones obtained from the pseudofunctoriality of $(-)^{\boxtimes}$. Then the following four diagrams of isomorphisms commute.

$$\begin{array}{ccccccc} & F^{\boxtimes} I^{\boxtimes} J^{\boxtimes} & & F'^{\boxtimes} J^{\boxtimes} I^{\boxtimes} & & I^{\boxtimes} J^{\boxtimes} G^{\boxtimes} & & J^{\boxtimes} I^{\boxtimes} G'^{\boxtimes} \\ \swarrow \text{via } \varphi & \downarrow \text{via } C_{F,I} & \swarrow \text{via } \psi & \downarrow \text{via } C_{F',J} & \swarrow \text{via } \varphi & \downarrow \text{via } C_{J,G} & \swarrow \text{via } \psi & \downarrow \text{via } C_{I,G'} \\ F^{\boxtimes} & \xleftarrow{C_{F',J}} F'^{\boxtimes} J^{\boxtimes} & \xleftarrow{C_{F,I}} F'^{\boxtimes} I^{\boxtimes} & \xleftarrow{C_{F',J}} F'^{\boxtimes} I^{\boxtimes} & \xleftarrow{C_{I,G'}} I^{\boxtimes} G'^{\boxtimes} & \xleftarrow{C_{J,G}} J^{\boxtimes} G'^{\boxtimes} & \xleftarrow{C_{J,G}} J^{\boxtimes} G'^{\boxtimes} & \xleftarrow{C_{I,G'}} J^{\boxtimes} G'^{\boxtimes} \end{array}$$

PROOF. All the four diagrams are handled in a similar way. For the first two diagrams from the left, we use vertical transitivity of base-change (Lemma 3.2.5(iii)) corresponding to the following two diagrams respectively.

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ (1_X, P) \parallel & & \downarrow I \\ X & \xrightarrow{F'} & Y' \\ (1_X, P) \parallel & & \downarrow J \\ X & \xrightarrow{F} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{F'} & Y' \\ (1_X, P) \parallel & & \downarrow J \\ X & \xrightarrow{F} & Y \\ (1_X, P) \parallel & & \downarrow I \\ X & \xrightarrow{F'} & Y' \end{array}$$

A similar argument works for the other two cases. \square

For the next lemma, the basic premise is that there is a cartesian square \mathfrak{s} drawn as follows where F and F_1 have label P .

$$\begin{array}{ccc} W & \xrightarrow{G_1} & X \\ F_1 \downarrow & & \downarrow F \\ Z & \xrightarrow{G} & Y \end{array}$$

We shall now consider cases where exactly one of its vertices is modified via a P -labeled isomorphism. In each case the new edges shall be called F' , G' , F'_1 or G'_1 as applicable and each of these new maps shall have the same label as that of its old counterpart. Also, in each case, the modified square shall be called \mathfrak{s}' .

LEMMA 3.5.4. *With notation and convention as above, the following hold.*

- (i) *Let $I: W \rightarrow W'$ and $J: W' \rightarrow W$ be \mathbf{P} -labeled inverse isomorphisms. Then the following diagram commutes.*

$$\begin{array}{ccc}
 G_1^\boxtimes F^\boxtimes & \xrightarrow{\beta_s} & F_1^\boxtimes G^\boxtimes \\
 \text{via } C_{I, G_1'}^{-1} \downarrow & & \downarrow \text{via } C_{I, F_1'}^{-1} \\
 I^\boxtimes G_1'^\boxtimes F^\boxtimes & \xrightarrow[\beta_{s'}]{\text{via}} & I^\boxtimes F_1'^\boxtimes G^\boxtimes
 \end{array}$$

- (ii) *Let $I: Z \rightarrow Z'$ and $J: Z' \rightarrow Z$ be \mathbf{P} -labeled inverse isomorphisms. Then the following diagram commutes.*

$$\begin{array}{ccccc}
 & & F_1^\boxtimes G^\boxtimes & & \\
 & \nearrow \beta_s^{-1} & & \searrow \text{via } C_{I, G'}^{-1} & \\
 G_1^\boxtimes F^\boxtimes & & F_1'^\boxtimes J^\boxtimes G^\boxtimes & & F_1^\boxtimes I^\boxtimes G'^\boxtimes \\
 & \nwarrow \text{via } C_{F_1', J} & & \nearrow \text{via } C_{J, G} & \\
 & & F_1'^\boxtimes G'^\boxtimes & & \\
 & \nwarrow \beta_{s'} & & \nearrow \text{via } C_{F_1, I}^{-1} &
 \end{array}$$

- (iii) *Let $I: X \rightarrow X'$ and $J: X' \rightarrow X$ be \mathbf{P} -labeled inverse isomorphisms. Then the following diagram commutes.*

$$\begin{array}{ccccc}
 & & G_1^\boxtimes F^\boxtimes & & \\
 & \nearrow \text{via } C_{G_1', J} & & \searrow \beta_s & \\
 G_1'^\boxtimes J^\boxtimes F^\boxtimes & & G_1^\boxtimes I^\boxtimes F'^\boxtimes & & F_1^\boxtimes G^\boxtimes \\
 & \nwarrow \text{via } C_{J, F} & & \nearrow \beta_{s'}^{-1} & \\
 & & G_1'^\boxtimes F'^\boxtimes & & \\
 & \nwarrow \text{via } C_{G_1, I}^{-1} & & \nearrow \text{via } C_{I, F'}^{-1} &
 \end{array}$$

- (iv) *Let $I: Y \rightarrow Y'$ and $J: Y' \rightarrow Y$ be \mathbf{P} -labeled inverse isomorphisms. Then the following diagram commutes.*

$$\begin{array}{ccc}
 G_1^\boxtimes F^\boxtimes & \xrightarrow{\beta_s} & F_1^\boxtimes G^\boxtimes \\
 \text{via } C_{F', J}^{-1} \downarrow & & \downarrow \text{via } C_{G', J}^{-1} \\
 G_1^\boxtimes F'^\boxtimes J^\boxtimes & \xrightarrow[\beta_{s'}]{\text{via}} & F_1^\boxtimes G'^\boxtimes J^\boxtimes
 \end{array}$$

PROOF. We only give proofs of (i) and (ii). The remaining two cases are handled similarly.

In (i), commutativity follows by transitivity of base-change associated to the following diagram where s' is the square at the bottom and s is the composite

square. (For \mathcal{C}_{I,F'_1} we also refer to part 2 of 3.5.2.)

$$\begin{array}{ccc}
 W & \xrightarrow{G_1} & X \\
 I \downarrow & & \parallel (1_X, \mathbb{P}) \\
 W' & \xrightarrow{G'_1} & X \\
 F'_1 \downarrow & & \downarrow F \\
 Z & \xrightarrow{G} & Y
 \end{array}$$

In (ii), the diagram has two parts. The non-convex part on the left commutes because of transitivity of base-change associated to the following diagram where \mathfrak{s}' is the square at the top and \mathfrak{s} is the composite square. (For $\mathcal{C}_{F'_1,J}$ we also refer to part 2 of 3.5.2.)

$$\begin{array}{ccc}
 W & \xrightarrow{G_1} & X \\
 F'_1 \downarrow & & \downarrow F \\
 Z' & \xrightarrow{G'} & Y \\
 J \downarrow & & \parallel (1_Y, \mathbb{P}) \\
 Z & \xrightarrow{G} & Y
 \end{array}$$

The rhombus on the right in (ii) is the outer border of the following diagram

$$\begin{array}{ccccc}
 & & F_1^\boxtimes G^\boxtimes & & \\
 & \nearrow \text{via } \mathcal{C}_{F'_1,J} & \uparrow \text{via } I^\boxtimes J^\boxtimes \cong 1 & \nwarrow \text{via } \mathcal{C}_{I,G'}^{-1} & \\
 F_1^\boxtimes J^\boxtimes G^\boxtimes & \xrightarrow{\text{via } \mathcal{C}_{F_1,I}^{-1}} & F_1^\boxtimes I^\boxtimes J^\boxtimes G^\boxtimes & \xrightarrow{\text{via } \mathcal{C}_{J,G}} & F_1^\boxtimes I^\boxtimes G'^\boxtimes \\
 & \searrow \text{via } \mathcal{C}_{J,G} & & \nearrow \text{via } \mathcal{C}_{F_1,I}^{-1} & \\
 & & F_1^\boxtimes G'^\boxtimes & &
 \end{array}$$

where the upper two triangles commute by 3.5.3 (see its first and third diagrams from the left) while the lower part commutes for functorial reasons. \square

We now consider some situations that arise in the proof of Proposition 3.5.9. In each case, one looks at a local piece of a staircase diagram. The goal is to describe the cumulative effect resulting from replacing a vertex by an isomorphic one. Here is the setup of the first situation.

Consider the following two cartesian diagrams, differing only via the middle vertex and satisfying conditions described below.

$$\begin{array}{ccccc}
 X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 X_4 & \longrightarrow & X_5 & \longrightarrow & X_6 \\
 \downarrow & & \downarrow & & \downarrow \\
 X_7 & \longrightarrow & X_8 & \longrightarrow & X_9
 \end{array}
 \qquad
 \begin{array}{ccccc}
 X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\
 \downarrow & & \downarrow & & \downarrow \\
 X_4 & \longrightarrow & X'_5 & \longrightarrow & X_6 \\
 \downarrow & & \downarrow & & \downarrow \\
 X_7 & \longrightarrow & X_8 & \longrightarrow & X_9
 \end{array}$$

The maps in the above diagrams are named as follows. For the diagram on the left, we use $F_{i,j}$ for the drawn map $X_i \rightarrow X_j$. For the diagram on the right, if $i \neq 5$ and $j \neq 5$ then the map $X_i \rightarrow X_j$ is the same as $F_{i,j}$ and for the remaining cases we use $F'_{i,j}$. Finally, we name the diagram on the left as \mathcal{S} and the one on the right as \mathcal{S}' .

The above two diagrams \mathcal{S} and \mathcal{S}' are assumed to further satisfy the following properties.

- The vertical maps are all \mathbf{P} -labeled.
- Whenever $F'_{i,j}$ exists, it has the same label that $F_{i,j}$ has.
- There are \mathbf{P} -labeled inverse isomorphisms $I: X_5 \rightarrow X'_5$ and $J: X'_5 \rightarrow X_5$ such that whenever $F'_{i,j}$ exists, $F_{i,j}$ and $F'_{i,j}$ are related via I, J in the obvious manner.

Using the base-change isomorphisms for each of the unit squares in \mathcal{S} and likewise for \mathcal{S}' , one obtains generalized base-change isomorphisms $\beta_{\mathcal{S}}$ and $\beta_{\mathcal{S}'}$ respectively, each of the following type

$$F_{1,2}^{\boxtimes} F_{2,3}^{\boxtimes} F_{3,6}^{\boxtimes} F_{6,9}^{\boxtimes} \xrightarrow{\sim} F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes}.$$

LEMMA 3.5.5. *In the above situation, $\beta_{\mathcal{S}} = \beta_{\mathcal{S}'}$.*

PROOF. (cf. Remark 3.5.6 below.) It suffices to prove that the following diagram commutes. Here the column of four maps on the left end defines $\beta_{\mathcal{S}}$ through the base-change isomorphisms associated to the four cartesian squares in \mathcal{S} . Likewise, the column at the right end defines $\beta_{\mathcal{S}'}$.

$$\begin{array}{ccccc}
F_{1,2}^{\boxtimes} F_{2,3}^{\boxtimes} F_{3,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \xlongequal{\quad} & & & F_{1,2}^{\boxtimes} F_{2,3}^{\boxtimes} F_{3,6}^{\boxtimes} F_{6,9}^{\boxtimes} \\
\downarrow & & \blacksquare_1 & & \downarrow \\
F_{1,2}^{\boxtimes} F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \xrightarrow{6} & F_{1,2}^{\boxtimes} F_{2,5}^{\boxtimes} I^{\boxtimes} F_{5,6}'^{\boxtimes} F_{6,9}^{\boxtimes} & \xrightarrow{2} & F_{1,2}^{\boxtimes} F_{2,5}'^{\boxtimes} F_{5,6}'^{\boxtimes} F_{6,9}^{\boxtimes} \\
\downarrow & & \blacksquare_2 & & \downarrow \\
F_{1,4}^{\boxtimes} F_{4,5}^{\boxtimes} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \xrightarrow{6} & F_{1,4}^{\boxtimes} F_{4,5}^{\boxtimes} I^{\boxtimes} F_{5,6}'^{\boxtimes} F_{6,9}^{\boxtimes} & \xrightarrow{4} & F_{1,4}^{\boxtimes} F_{4,5}'^{\boxtimes} F_{5,6}'^{\boxtimes} F_{6,9}^{\boxtimes} \\
\downarrow & & \blacksquare_3 & & \downarrow \\
F_{1,4}^{\boxtimes} F_{4,5}^{\boxtimes} F_{5,8}^{\boxtimes} F_{8,9}^{\boxtimes} & \xrightarrow{8} & F_{1,4}^{\boxtimes} F_{4,5}^{\boxtimes} I^{\boxtimes} F_{5,8}'^{\boxtimes} F_{8,9}^{\boxtimes} & \xrightarrow{4} & F_{1,4}^{\boxtimes} F_{4,5}'^{\boxtimes} F_{5,8}'^{\boxtimes} F_{8,9}^{\boxtimes} \\
\downarrow & & & & \downarrow \\
F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes} & \xlongequal{\quad} & & & F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes}
\end{array}$$

The horizontal maps marked as 6 (resp. 2, 4, 8) are the obvious ones induced by $C_{I, F_{5,6}}^{-1}$ (resp. $C_{F_{2,5}, I}$, $C_{F_{4,5}, I}$, $C_{I, F_{5,8}}^{-1}$). The vertical maps are all induced by β_- with an obvious choice for the subscript in each case (corresponding to one of the unit squares in \mathcal{S} or \mathcal{S}').

Commutativity of all the rectangles are proved using 3.5.4 or functoriality considerations. We concentrate only on \blacksquare_1 , \blacksquare_2 and \blacksquare_3 by way of illustration.

In \blacksquare_1 , we may cancel the common factors $F_{1,2}^{\boxtimes}$ on the left and $F_{6,9}^{\boxtimes}$ on the right from each vertex, and so we reduce to checking commutativity of the following

diagram.

$$\begin{array}{ccc} F_{2,3}^{\boxtimes} F_{3,6}^{\boxtimes} & \longrightarrow & F_{2,5}'^{\boxtimes} F_{5,6}'^{\boxtimes} \\ \downarrow & & \downarrow \\ F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes} & \longrightarrow & F_{2,5}^{\boxtimes} I^{\boxtimes} F_{5,6}'^{\boxtimes} \end{array}$$

This diagram commutes by 3.5.4(ii) using $G_1 = F_{2,3}, F_1 = F_{2,5}, F = F_{3,6}, G = F_{5,6}$ and $F_1' = F_{2,5}', G' = F_{5,6}'$.

In \blacksquare_3 , we reduce to checking commutativity of the following diagram

$$\begin{array}{ccc} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \longrightarrow & I^{\boxtimes} F_{5,6}'^{\boxtimes} F_{6,9}^{\boxtimes} \\ \downarrow & & \downarrow \\ F_{5,8}^{\boxtimes} F_{8,9}^{\boxtimes} & \longrightarrow & I^{\boxtimes} F_{5,8}'^{\boxtimes} F_{8,9}^{\boxtimes} \end{array}$$

and we conclude by 3.5.4(i) using $G_1 = F_{5,6}, F_1 = F_{5,8}, F = F_{6,9}, G = F_{8,9}$ and $F_1' = F_{5,8}', G_1' = F_{8,9}'$.

Finally, \blacksquare_2 commutes for functorial reasons. \square

REMARK 3.5.6. We have given a longer proof above, because a good portion of it is used for tackling 3.5.7 below. A quicker way to prove 3.5.5 would be to note that since all the vertical maps in \mathcal{S} and \mathcal{S}' have the same label (which is P), one can use vertical transitivity of base-change for any pair of squares stacked vertically. Now all the vertical composites of \mathcal{S} and \mathcal{S}' , including the ones in the middle, viz., $|F_{2,5} \star F_{5,8}|$ and $|F_{2,5}' \star F_{5,8}'|$, are equal. Thus 3.5.5 follows.

The next situation involves the following two cartesian diagrams \mathcal{S} and \mathcal{S}' differing only at the middle vertex.

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & & \\ \downarrow & & \downarrow & & \\ X_4 & \longrightarrow & X_5 & \longrightarrow & X_6 \\ \downarrow & & \downarrow & & \downarrow \\ X_7 & \longrightarrow & X_8 & \longrightarrow & X_9 \end{array} \quad \begin{array}{ccccc} X_1 & \longrightarrow & X_2 & & \\ \downarrow & & \downarrow & & \\ X_4 & \longrightarrow & X_5' & \longrightarrow & X_6 \\ \downarrow & & \downarrow & & \downarrow \\ X_7 & \longrightarrow & X_8 & \longrightarrow & X_9 \end{array}$$

We adopt the same convention for naming the maps as before. The properties satisfied by \mathcal{S} and \mathcal{S}' are the same as before except for the following additional condition: We assume that $X_2 = X_6$ (each being henceforth given the common name X), and that $|F_{2,5} \star F_{5,6}| = 1_X = |F_{2,5}' \star F_{5,6}'|$.

Under these assumptions, we obtain two natural candidates for an isomorphism of the following form, one via the cartesian squares of \mathcal{S} and the fundamental isomorphism $\phi_{F_{2,5}, F_{5,6}}$, and the other via $\mathcal{S}' \dots$

$$F_{1,2}^{\boxtimes} F_{6,9}^{\boxtimes} = F_{1,2}^{\boxtimes} \mathbf{1}_{\mathcal{D}_X} F_{6,9}^{\boxtimes} \xrightarrow{\sim} F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes}$$

LEMMA 3.5.7. *The above-mentioned two natural candidates are the same.*

PROOF. It suffices to check commutativity of the following replacement of the diagram in the proof of 3.5.5. The topmost rectangle there, \blacksquare_1 , has been replaced

by $\tilde{\blacksquare}$. The remaining portion remains the same and only its outer border is shown below.

$$\begin{array}{ccc}
 F_{1,2}^{\boxtimes} F_{6,9}^{\boxtimes} & \xlongequal{\quad} & F_{1,2}^{\boxtimes} F_{6,9}^{\boxtimes} \\
 \downarrow & \tilde{\blacksquare} & \downarrow \\
 F_{1,2}^{\boxtimes} F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \xrightarrow{6} F_{1,2}^{\boxtimes} F_{2,5}^{\boxtimes} I^{\boxtimes} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} \xrightarrow{2} F_{1,2}^{\boxtimes} F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes} F_{6,9}^{\boxtimes} & \\
 \downarrow & \text{use lower part of diagram in proof of 3.5.5} & \downarrow \\
 F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes} & \xlongequal{\quad} & F_{1,4}^{\boxtimes} F_{4,7}^{\boxtimes} F_{7,8}^{\boxtimes} F_{8,9}^{\boxtimes}
 \end{array}$$

Commutativity of $\tilde{\blacksquare}$, after the canceling of the common factor $F_{1,2}^{\boxtimes} F_{6,9}^{\boxtimes}$ from each vertex, reduces to checking that of the following.

$$\begin{array}{ccccc}
 & & \mathbf{1}_{\mathcal{D}_X} & \xlongequal{\quad} & \mathbf{1}_{\mathcal{D}_X} \\
 & \uparrow & & & \uparrow \\
 F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes} & \longrightarrow & F_{2,5}^{\boxtimes} I^{\boxtimes} F_{5,6}^{\boxtimes} & \longrightarrow & F_{2,5}^{\boxtimes} F_{5,6}^{\boxtimes}
 \end{array}$$

This diagram commutes because of compatibility of the fundamental isomorphism with base-change (Lemma 3.2.5(ii)) in the context of the following diagram. (Recall that $X = X_2 = X_6$.)

$$\begin{array}{ccccc}
 X_2 & \longrightarrow & X_5 & \longrightarrow & X_6 \\
 \parallel & & \downarrow I & & \parallel \\
 X_2 & \longrightarrow & X'_5 & \longrightarrow & X_6
 \end{array}$$

□

For the next two situations we look at truncated versions of diagrams encountered in the previous two results. Consider the following two conjugate pairs of cartesian diagrams.

$$\begin{array}{ccc}
 \begin{array}{ccc} X_2 & & X_2 \\ \downarrow & & \downarrow \\ X_5 & \longrightarrow & X_6 \\ \downarrow & & \downarrow \\ X_8 & \longrightarrow & X_9 \end{array} &
 \begin{array}{ccc} X_2 & & X_2 \\ \downarrow & & \downarrow \\ X'_5 & \longrightarrow & X_6 \\ \downarrow & & \downarrow \\ X_8 & \longrightarrow & X_9 \end{array} &
 \begin{array}{ccc} X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow \\ X_5 & \longrightarrow & X_6 \\ \downarrow & & \downarrow \\ X_8 & \longrightarrow & X_9 \end{array} \quad
 \begin{array}{ccc} X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow \\ X'_5 & \longrightarrow & X_6 \\ \downarrow & & \downarrow \\ X_8 & \longrightarrow & X_9 \end{array}
 \end{array}$$

Let us call the diagrams $\mathcal{S}_1, \mathcal{S}'_1, \mathcal{S}_2, \mathcal{S}'_2$ respectively starting from the left. We use $F_{i,j}, F'_{i,j}$ for maps in $\mathcal{S}_1, \mathcal{S}'_1$ and $G_{i,j}, G'_{i,j}$ for the ones in $\mathcal{S}_2, \mathcal{S}'_2$. In $\mathcal{S}_1, \mathcal{S}'_1$, the assumptions are the same as that from 3.5.7, (in particular, $X_2 = X_6 = X$, etc. ...) while in $\mathcal{S}_2, \mathcal{S}'_2$ we use the ones from 3.5.5.

One point of departure from the earlier cases is that here we shall also look at the composite vertical maps. Thus $F_{2,8}, G_{3,9}$ and $G_{2,8} = G'_{2,8}$ are the obvious composite maps with label P.

Proceeding as before, the above diagrams lead to two conjugate natural candidates each for isomorphisms of the following form.

$$F_{6,9}^{\boxtimes} = \mathbf{1}_{\mathcal{D}_X} F_{6,9}^{\boxtimes} \xrightarrow{\sim} F_{2,8}^{\boxtimes} F_{8,9}^{\boxtimes} \quad G_{2,3}^{\boxtimes} G_{3,9}^{\boxtimes} \xrightarrow{\sim} G_{2,8}^{\boxtimes} G_{8,9}^{\boxtimes}$$

LEMMA 3.5.8. *In each of the two cases above, the conjugate isomorphisms are the same.*

PROOF. See proofs of 3.5.5 and 3.5.7. Also see Remark 3.5.6. \square

We are finally in a position to prove the main result of this subsection.

PROPOSITION 3.5.9. *The fundamental isomorphism of (3.4.1.5) is canonical, i.e., does not depend on the choice of the staircase.*

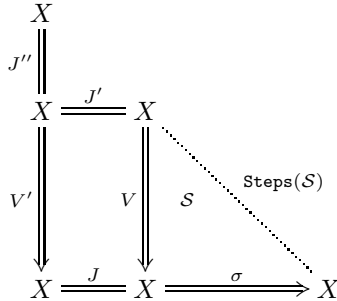
PROOF. Suppose \mathcal{S} and \mathcal{S}' are two staircase diagrams based on σ . We proceed by successively modifying \mathcal{S} , one vertex at a time, till it is transformed into \mathcal{S}' . The starting point for these modifications is the southeastern end of the staircase. Suppose \mathcal{S} is represented by the diagram in (3.4.1.2) and the conjugate staircase \mathcal{S}' is represented by a corresponding $(-)'$ version. Then we consider the vertex $Y_{n,n-1}$ of \mathcal{S} and replace it by $Y'_{n,n-1}$, while also modifying the arrows coming in and out of it via the unique isomorphism $Y_{n,n-1} \cong Y'_{n,n-1}$ arising out of the fiber-product property. If $n > 2$, then by 3.5.7 we see that both \mathcal{S} and the newly modified staircase give rise to the same isomorphism and if $n = 2$ we use 3.5.8 for the same conclusion. Thus, without loss of generality, we may assume that $Y_{n,n-1}$ and the arrows going out of it are all shared by \mathcal{S} and \mathcal{S}' . (This automatically forces $V_{n-1,n-1} = V'_{n-1,n-1}$ too.) Next, we replace $Y_{n,n-2}$. Arguing the same way as above we continue westward till the bottom row is assumed to be shared by \mathcal{S} and \mathcal{S}' . Then we start with $Y_{n-1,n-2}$. Continuing in this fashion, the proposition is proved. \square

DEFINITION 3.5.10. For any sequence σ such that $|\sigma|$ is an identity map, we define Φ_σ to be $\Phi_\sigma(\mathcal{S})$ of (3.4.1.5) for any choice of a staircase \mathcal{S} based on σ .

The following lemma is used in §5.

LEMMA 3.5.11. *Let $\sigma: X \Rightarrow X$ be a sequence such that $|\sigma| = 1_X$ and set $J := (1_X, \mathbf{P})$. Then via the identification $J^{\boxtimes} \sigma^{\boxtimes} = \sigma^{\boxtimes}$, we have $\Phi_{J \star \sigma} = \Phi_\sigma$.*

PROOF. Let \mathcal{S} be a staircase based on σ . Then a staircase based on $J \star \sigma$ can be obtained as shown below where $V = V_1(\mathcal{S})$ is the leftmost column of \mathcal{S} and $V' = V, J'' = J' = J$.



By pseudofunctoriality of $(-)^{\times}$ we conclude that the isomorphism θ given by composing the following ones

$$\mathbf{1}_{\mathcal{D}_X} = J^{\boxtimes} \xrightarrow{\sim} J''^{\boxtimes} V'^{\boxtimes} J^{\boxtimes} \xrightarrow{\sim} J''^{\boxtimes} J'^{\boxtimes} V^{\boxtimes} \xrightarrow{\sim} V^{\boxtimes}$$

is the same as the obvious one resulting from composing all the (necessarily P-labeled) maps in V . The desired result now follows by comparing the definition of Φ_{σ} with that of $\Phi_{J\star\sigma}$. \square

4. Proofs II (the cocycle condition)

This section contains some of the core results used in the proof of the abstract output results of the paper. We begin with the definition of $\Psi_{-, -}$ that gives an isomorphism $\sigma_1^{\boxtimes} \xrightarrow{\sim} \sigma_2^{\boxtimes}$ whenever $|\sigma_1| = |\sigma_2|$. Then we show that $\Psi_{-, -}$ satisfies the cocycle condition. The proof of this is somewhat long and occupies most of this section. We reduce the proof into a few lemmas, which are then verified over different subsections.

4.1. The definition of $\Psi_{-, -}$. The isomorphism $\Psi_{-, -}$ that we define here is easy to obtain using results defined from the previous section. Its canonicity however will not be immediately obvious owing to dependence on certain fibered-product diagrams needed in the construction. It is only after proving the cocycle condition, the topic of the next few subsections, that we get to conclude canonicity.

4.1.1. Let σ_1, σ_2 be two labeled sequences such that $|\sigma_1| = |\sigma_2|$. Then we want to define an isomorphism

$$\Psi_{\sigma_1, \sigma_2} : \sigma_2^{\boxtimes} \xrightarrow{\sim} \sigma_1^{\boxtimes}.$$

This is achieved as follows. Let X be the source of σ_i and Y the target. Consider the following diagram where the square \mathcal{S} of double arrows is cartesian (§3.3.1) and δ is the diagonal map, which, by §2.1[A](ii), is in P.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta=(\delta, P)} & X \times_Y X & \xrightleftharpoons{\sigma'_2} & X \\ & & \sigma'_1 \downarrow & \mathcal{S} & \downarrow \sigma_1 \\ & & X & \xrightleftharpoons{\sigma_2} & Y \end{array}$$

We define $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ to be the composition of the following sequence of isomorphisms.

$$\begin{aligned} (4.1.1.1) \quad \sigma_2^{\boxtimes} & \xrightarrow[\text{3.5.10}]{\text{via } (\Phi_{\Delta \star \sigma'_1})^{-1}} (\Delta \star \sigma'_1)^{\boxtimes} \sigma_2^{\boxtimes} = \Delta^{\boxtimes} (\sigma'_1 \star \sigma_2)^{\boxtimes} \\ & \cong \Delta^{\boxtimes} (\sigma'_2 \star \sigma_1)^{\boxtimes} \quad (\text{via } \beta_{\mathcal{S}}^{-1}) \\ & = (\Delta \star \sigma'_2)^{\boxtimes} \sigma_1^{\boxtimes} \xrightarrow[\text{3.5.10}]{\text{via } \Phi_{\Delta \star \sigma'_2}} \sigma_1^{\boxtimes} \end{aligned}$$

REMARK 4.1.2. As in the case of the generalized fundamental isomorphism, we would like to say that $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ does not depend on the choice of \mathcal{S} which is determined only up to isomorphism. However, the approach that we took in §3.5 cannot be adapted here because there we exploited the fact that vertical maps of a staircase are all P-labeled. What is perhaps needed is a more elaborate set of compatibilities over the category \mathbf{I} of isomorphisms in \mathbf{C} . Instead, we resort to the following strategy. We show that the cocycle condition for $\Psi_{-, -}$, a property that we are interested in for various reasons, holds even at the level of the extrinsically

defined $\Psi_{-, -}^-$. Then, using the cocycle condition we deduce that the definition is indeed canonical. The price we pay for this indirect approach is of having to retain the cumbersome notation such as $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ instead of $\Psi_{\sigma_1, \sigma_2}$, and in particular, of keeping track of the choice of \mathcal{S} whenever necessary, till the end of the proof of the cocycle condition.

4.1.3. The following particular cases of (4.1.1.1) are noteworthy.

(i) Let σ be a sequence such that $f := |\sigma| \in \mathbf{P}$. Set $F := (f, \mathbf{P})$. Then one obtains an isomorphism (for any available choice of \mathcal{S}) $\Psi_{F, \sigma}^{\mathcal{S}}: \sigma^{\boxtimes} \xrightarrow{\sim} f^{\times}$.

(ii) Similarly, if we assume instead, that $f = |\sigma| \in \mathbf{O}$, then, with $F := (f, \mathbf{O})$, we obtain an isomorphism $\Psi_{F, \sigma}^{\mathcal{S}}: \sigma^{\boxtimes} \xrightarrow{\sim} f^{\square}$.

It can be shown that $\Psi_{-, -}^-$ recovers the fundamental isomorphism, in that if we let f in (i) above to be an identity map, then for a suitable choice of \mathcal{S} , (and hence for any choice) the isomorphism in (i) gives the fundamental isomorphism Φ_{σ} . However, we postpone the verification of this fact to the next section.

4.2. The cocycle condition. Perhaps the most important result lying at the technical heart of this paper is the following.

THEOREM 4.2.1. *The isomorphism $\Psi_{-, -}^-$ of (4.1.1.1) satisfies the cocycle condition, i.e., if $\sigma_1, \sigma_2, \sigma_3$ are labeled sequences such that $|\sigma_1| = |\sigma_2| = |\sigma_3|$, then the following condition holds*

$$\Psi_{\sigma_3, \sigma_2}^{\mathcal{T}} \Psi_{\sigma_2, \sigma_1}^{\mathcal{S}} = \Psi_{\sigma_3, \sigma_1}^{\mathcal{U}}$$

for any available choice of fibered product diagrams $\mathcal{S}, \mathcal{T}, \mathcal{U}$.

REMARK 4.2.2. We use the cocycle condition as stated above in two stages. In the first one, the presence of superscripts of Ψ is important. This is really catered towards proving that $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ of (4.1.1.1) is independent of the choice of \mathcal{S} , which would then give us a canonical definition of $\Psi_{\sigma_1, \sigma_2}$ (4.6.2). Once this is accomplished, so that we have the luxury of dropping the superscripts for Ψ , the true cocycle condition emerges. The trimmed version leads to many interesting consequences including the existence of a pseudofunctor over $\mathbf{Q} = \overline{\{\mathbf{O}, \mathbf{P}\}}$ that generalizes $(-)^{\square}$ and $(-)^{\times}$.

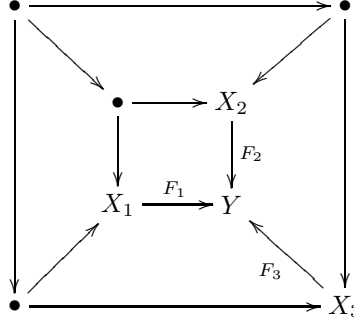
Before going into the proof of 4.2.1, we make some general remarks on fibered products.

4.2.3. For the remainder of this paper, we resort to the following somewhat ambiguous convention. For any cartesian square $\mathfrak{s} := (F_1, F_2, F'_1, F'_2)$ we use β_{F_1, F_2} in place of $\beta_{\mathfrak{s}}$. To eliminate any ambiguity, this notation is employed only in situations where all the components of \mathfrak{s} are already chosen unambiguously. Note that under such circumstances we may write $\beta_{F_1, F_2} = \beta_{F_2, F_1}^{-1}$.

A similar convention also applies in the more general situation of products of sequences. Thus given two sequences σ_1, σ_2 having the same target and given a choice of a fibered square \mathcal{S} on (σ_1, σ_2) , we use $\beta_{\sigma_1, \sigma_2}$ (resp. $\beta_{\sigma_2, \sigma_1}$) to denote $\beta_{\mathcal{S}}$ (resp. $\beta_{\mathcal{S}}^{-1}$).

4.2.4. Given three maps $X_i \xrightarrow{F_i} Y$ for $i = 1, 2, 3$, a cartesian cube (or fibered cube) on F_1, F_2 and F_3 has the obvious meaning (each of the six faces is a cartesian square) and is drawn as follows where the symbol \bullet stands for the appropriate

fibered products.



Unlike the case of cartesian squares, here we avoid imposing an ordering on the edges since our interest is mainly on base-change isomorphisms associated to squares, for which the convention in 4.2.3 above suffices.

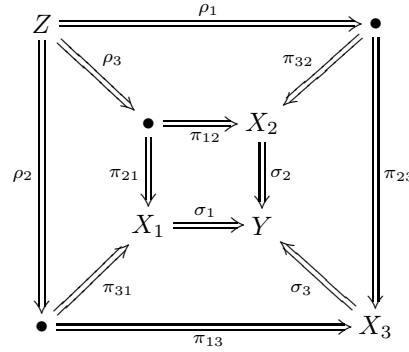
The existence of such a cartesian cube on F_1, F_2 and F_3 is a formal exercise on fibered products. In fact, the following also holds. For any choice of cartesian squares \mathfrak{s}_{12} (resp. \mathfrak{s}_{13} , resp. \mathfrak{s}_{23}) on F_1, F_2 (resp. F_1, F_3 , resp. F_2, F_3) there is a cartesian cube on F_1, F_2 and F_3 which has \mathfrak{s}_{12} , \mathfrak{s}_{13} and \mathfrak{s}_{23} for three of its faces.

The notion of a cartesian cube also generalizes to sequences. Thus, given three sequences $\sigma_i: X_i \rightarrow Y$ for $i = 1, 2, 3$, a cartesian cube on σ_1, σ_2 and σ_3 has the obvious meaning. Moreover, for any choice of squares $\mathfrak{s}_{1,2}, \mathfrak{s}_{1,3}, \mathfrak{s}_{2,3}$ obtained as fibered product of σ_1 and σ_2 , etc., there is a cartesian cube having these squares for three of its faces. As in the case of cartesian squares over sequences, we usually resort to drawing a cartesian cube only through its outermost edges, each represented by a double arrow.

Coming back to the topic of proving Theorem 4.2.1, we now state the three lemmas which are used in it.

We call the first of the following lemmas as the cube lemma.

LEMMA 4.2.5. *Let $\sigma_1, \sigma_2, \sigma_3$ be labeled sequences having the same target. Let X_i be the source of σ_i and Y the target. Consider a cartesian cube on σ_1, σ_2 and σ_3 as follows where Z is the product of X_1, X_2, X_3 over Y .*



Then the following hexagon of isomorphisms commutes. Here each object corresponds to one of the six different paths from Z to Y in the above diagram. For the

morphisms the omitted subscripts of the β 's are the obvious ones.

$$\begin{array}{ccccc}
 & & \rho_3^{\boxtimes} \pi_{21}^{\boxtimes} \sigma_1^{\boxtimes} & \xrightarrow{\rho_3^{\boxtimes}(\beta)} & \rho_3^{\boxtimes} \pi_{12}^{\boxtimes} \sigma_2^{\boxtimes} \\
 & \nearrow \beta(\sigma_1^{\boxtimes}) & & & \searrow \beta(\sigma_2^{\boxtimes}) \\
 \rho_2^{\boxtimes} \pi_{31}^{\boxtimes} \sigma_1^{\boxtimes} & & & & \rho_1^{\boxtimes} \pi_{32}^{\boxtimes} \sigma_2^{\boxtimes} \\
 & \searrow \rho_2^{\boxtimes}(\beta) & \rho_2^{\boxtimes} \pi_{13}^{\boxtimes} \sigma_3^{\boxtimes} & \xrightarrow{\beta(\sigma_3^{\boxtimes})} & \rho_1^{\boxtimes} \pi_{23}^{\boxtimes} \sigma_3^{\boxtimes} \\
 & & & \nearrow \rho_1^{\boxtimes}(\beta) &
 \end{array}$$

LEMMA 4.2.6. Let $X \xRightarrow{\sigma_1} Y \xRightarrow{\sigma_2} X$ be sequences such that $|\sigma_2 \star \sigma_1| = 1_X$, so that, by [A](i) and [A](ii) of §2.1, $|\sigma_1| \in \mathbf{P}$. Set $G := (|\sigma_1|, \mathbf{P})$. Then the following diagram of isomorphisms commutes for any available choice of \mathcal{S} .

$$\begin{array}{ccc}
 \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} & \xrightarrow{\Psi_{G, \sigma_1}^{\mathcal{S}}(\sigma_2^{\boxtimes})} & G^{\boxtimes} \sigma_2^{\boxtimes} \\
 \searrow \Phi_{\sigma_1 \star \sigma_2} & & \swarrow \Phi_{G \star \sigma_2} \\
 & \mathbf{1}_{\mathcal{D}_X} &
 \end{array}$$

LEMMA 4.2.7. Consider the following diagram

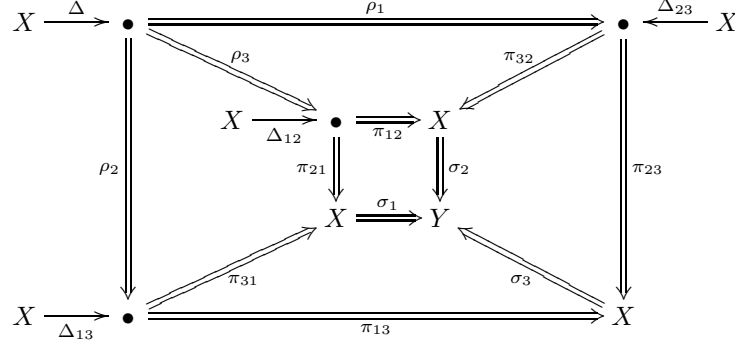
$$\begin{array}{ccccc}
 X & \xRightarrow{\sigma_1} & A & \xRightarrow{\sigma_2} & B \\
 & & \sigma_3 \downarrow & & \downarrow \sigma_4 \\
 & & C & \xRightarrow{\sigma_5} & D \xRightarrow{\sigma_6} X
 \end{array}$$

where the square is cartesian and $|\sigma_1 \star \sigma_3 \star \sigma_5 \star \sigma_6| = 1_X$. Then the following diagram of isomorphisms commutes where the Φ 's have the obvious subscripts.

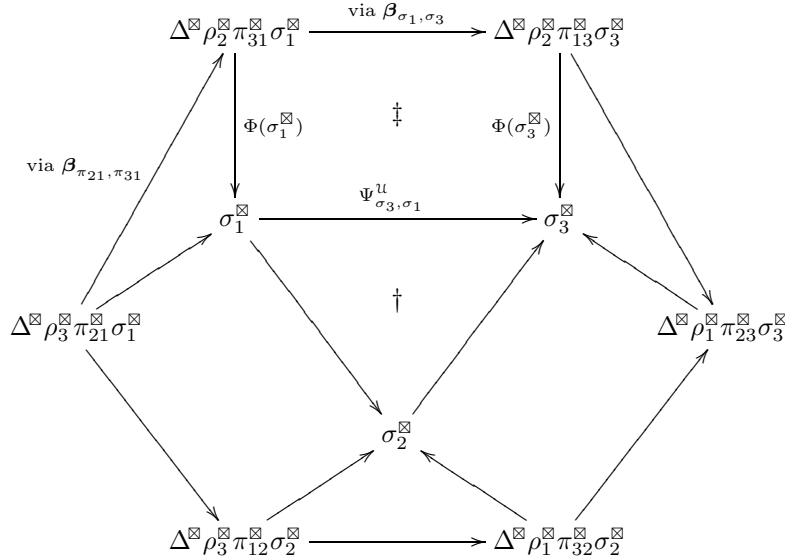
$$\begin{array}{ccc}
 \sigma_1^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} \sigma_6^{\boxtimes} & \xrightarrow{\text{via } \beta_{\sigma_5, \sigma_4}} & \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} \sigma_6^{\boxtimes} \\
 \searrow \Phi & & \swarrow \Phi \\
 & \mathbf{1}_{\mathcal{D}_X} &
 \end{array}$$

4.2.8. Assuming 4.2.5, 4.2.6 and 4.2.7, one proves 4.2.1 as follows. Let X be the source of σ_i and Y the target. Consider the following diagram where the cube of double arrows is cartesian having $\mathcal{S}, \mathcal{T}, \mathcal{U}$ as three of its faces in the obvious places.

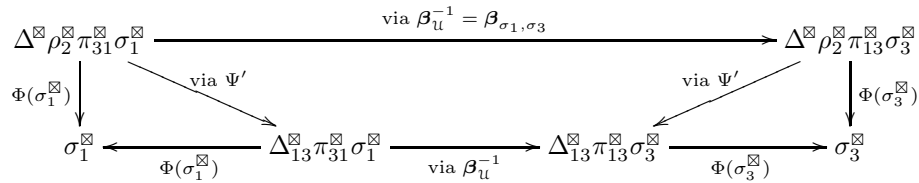
Also, Δ and Δ_{ij} are the appropriate diagonal maps with label P.



Now consider the following induced diagram of isomorphisms. To avoid clutter, we only describe the maps in \ddagger , the rest being defined analogously. In \ddagger , for the downward arrow on the left, Φ has subscript $\Delta \star \rho_2 \star \pi_{31}$, while for the one on the right, the subscript is $\Delta \star \rho_2 \star \pi_{13}$. The remaining definitions are self-evident.



The outer border, which is a hexagon, commutes, as is seen by first canceling Δ^\boxtimes from each vertex and then using the cube lemma (4.2.5). The three outer triangles commute by 4.2.7. For the rectangles we use 4.2.6. For example, let us consider \ddagger . We expand it as follows. The bottom row spells out the definition of $\Psi_{\sigma_3, \sigma_1}^u$. The Φ 's have the obvious subscripts and we use $\Psi' := \Psi_{\Delta_{13}, \Delta \star \rho_2}^v$ for some fixed available choice of \mathcal{V} .



The trapezium in the middle commutes for functorial reasons. The triangle on the left, upon canceling of the common term σ_1^\boxtimes from each vertex, results in the following one whose commutativity now follows from 4.2.6.

$$\begin{array}{ccc}
 \Delta^\boxtimes \rho_2^\boxtimes \pi_{31}^\boxtimes & & \\
 \downarrow \Phi(\Delta \star \rho_2) \star \pi_{31} & \searrow \text{via } \Psi_{\Delta_{13}, \Delta \star \rho_2}^\vee & \\
 1_{\mathcal{D}_X} & \xleftarrow{\Phi_{\Delta_{13} \star \pi_{31}}} & \Delta_{13}^\boxtimes \pi_{31}^\boxtimes
 \end{array}$$

A similar argument works for the triangle on the right. Thus \ddagger commutes. A similar proof works for the remaining rectangles.

It now follows that \dagger also commutes. This proves the cocycle condition modulo Lemmas 4.2.5, 4.2.6 and 4.2.7.

4.3. Proof of the cube lemma. This subsection is devoted to the proof of Lemma 4.2.5.

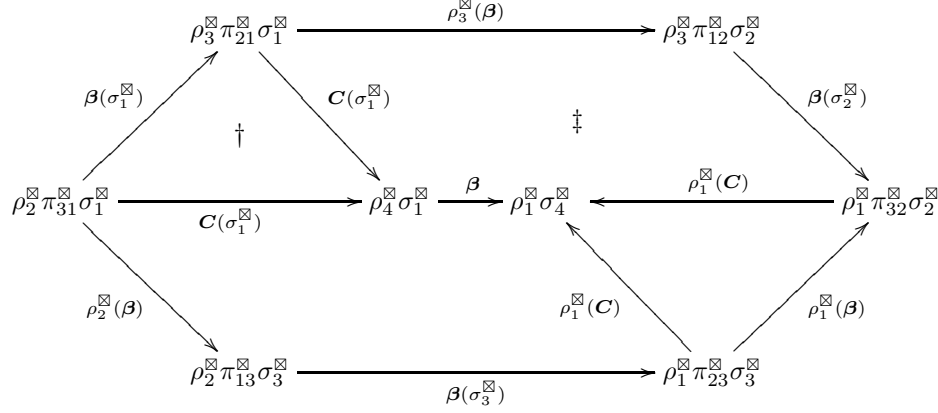
We proceed by induction on the lengths of the σ_i . To be precise, we use the sum of the lengths for induction. The proof is divided into two parts. The first concerns the basis of induction and the second, the inductive step.

4.3.1. The basis of induction is the case when σ_1, σ_2 and σ_3 , all are assumed to have length one. In this case we use the single-arrow notation for drawing the σ_i 's. Note that two of the σ_i 's must have the same label. Let us assume, without loss of generality, that σ_2 and σ_3 have the same label. Let X_4 be the fibered product of X_2 and X_3 over Y and let σ_4 be the induced map $X_4 \rightarrow Y$, having the same label as that of σ_2 . (Note that X_4 and the projection maps π_{32}, π_{23} are assumed to have been specified at the outset as per the hypothesis of 4.2.5.) Consider the following diagram where ρ_4 is the natural composite map $Z \rightarrow X_1$ having the same label as that of ρ_2 . Note that $(\sigma_4, \sigma_1, \rho_4, \rho_1)$ is also a cartesian square.

$$\begin{array}{ccccc}
 Z & \xrightarrow{\rho_1} & & & X_4 \\
 \downarrow \rho_2 & \searrow \rho_3 & & \swarrow \pi_{32} & \downarrow \pi_{23} \\
 & \bullet & \xrightarrow{\pi_{12}} & X_2 & \\
 & \searrow \rho_4 & \downarrow \pi_{21} & \downarrow \sigma_2 & \swarrow \sigma_4 \\
 & & X_1 & \xrightarrow{\sigma_1} & Y \\
 & \swarrow \pi_{31} & & \swarrow \sigma_3 & \\
 \bullet & \xrightarrow{\pi_{13}} & & & X_3
 \end{array}$$

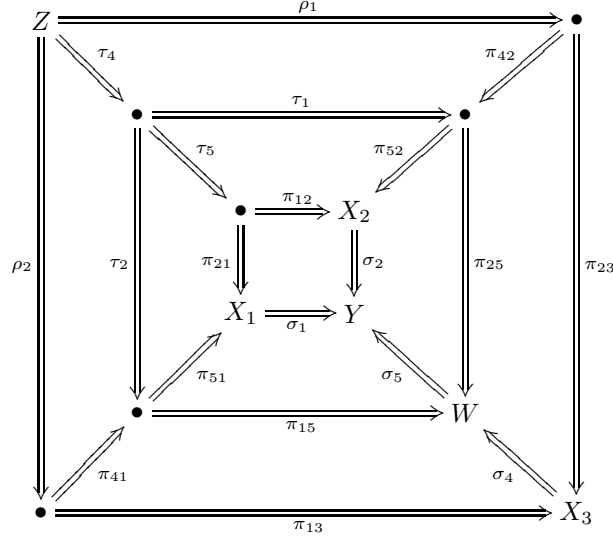
Consider the following induced diagram of isomorphisms where each object corresponds to one of the eight different paths from Z to Y and the omitted subscripts

of the β 's and the C 's (3.2.2) are the obvious ones.



Since σ_2, σ_3 and σ_4 have the same labels, by construction, this label is also shared by $\pi_{23}, \pi_{32}, \pi_{31}, \rho_2, \rho_3, \rho_4$. Therefore, \dagger , upon the canceling of the common term σ_1^\boxtimes from each vertex, commutes for pseudofunctorial reasons. For \ddagger , we deduce commutativity using the transitivity property of the β 's (3.2.5(iii)). The remaining subdiagrams above commute analogously. From the outer border of the preceding diagram, we therefore conclude that the cube lemma holds in the case when all the σ_i 's have length one.

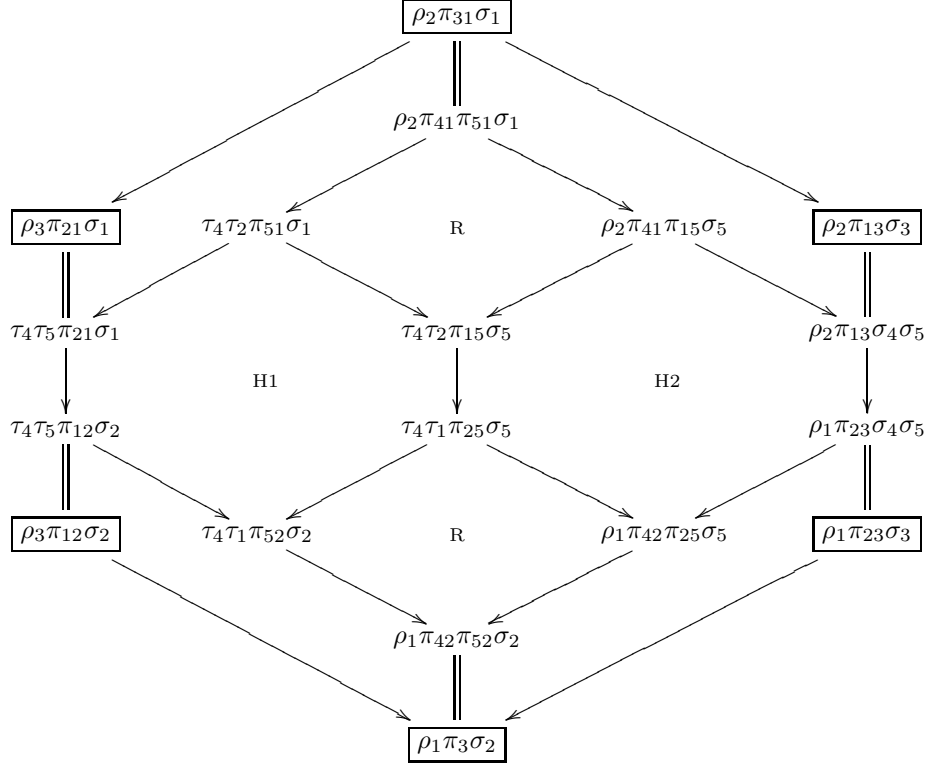
4.3.2. Now consider the case where one of the σ_i 's has length greater than one so that we may decompose it as a concatenation of two sequences; without loss of generality, let us assume that σ_3 factors as $X_3 \xrightarrow{\sigma_4} W \xrightarrow{\sigma_5} Y$. The corresponding cartesian diagram, a double cube as shown below, is the decomposition of the original cube into two cubes. (Once again, recall that all the faces of this double cube are assumed to have been already specified by hypothesis.)



Note that the following hold.

$$\rho_3 = \tau_4 \star \tau_5, \quad \pi_{31} = \pi_{41} \star \pi_{51}, \quad \pi_{32} = \pi_{42} \star \pi_{52}, \quad \sigma_3 = \sigma_4 \star \sigma_5$$

Consider the following diagram of isomorphisms where each unframed object corresponds to one of the twelve different paths from Z to Y in the preceding diagram. For convenience, we have dropped the common superscript \boxtimes from each term. The morphisms are determined using the following thumb rule: to determine any drawn morphism between two objects, first cancel the common terms from the two objects involved. The remaining terms then uniquely determine a base-change isomorphism $\beta_{-, -}$. The morphism under consideration is then the obvious one induced by $\beta_{-, -}$.



The two rhombuses, denoted by R, commute for functorial reasons. The two hexagons, H1 and H2, may be assumed to commute by the induction hypothesis; for H1 we use the cube constructed from π_{15} , π_{25} and σ_4 , while for H2 we use the cube constructed from σ_1 , σ_2 and σ_5 . The remaining four outer subdiagrams commute by definition of the base-change isomorphism (cf. 3.3.5); the equalities in these subdiagrams are obtained by comparing the double cube with the original one.

The outer border of the preceding diagram comprising the framed vertices and the induced maps now proves commutativity of the hexagon corresponding to the original cube.

By induction 4.2.5 follows.

4.4. Blocks of staircase. We give a proof of Lemma 4.2.6 below.

4.4.1. Let us recall 4.2.6 for convenience. Let $X \xrightarrow{\sigma_1} Y \xrightarrow{\sigma_2} X$ be sequences such that $|\sigma_1 \star \sigma_2| = 1_X$. Then with $G := (|\sigma_1|, \mathbf{P})$, the following diagram of isomorphisms commutes for any available choice of \mathcal{S} .

$$\begin{array}{ccc} \sigma_1^\boxtimes \sigma_2^\boxtimes & \xrightarrow{\Psi_{G, \sigma_1}^{\mathcal{S}}(\sigma_2^\boxtimes)} & G^\boxtimes \sigma_2^\boxtimes \\ & \searrow \Phi_{\sigma_1 \star \sigma_2} \quad \swarrow \Phi_{G \star \sigma_2} & \\ & \mathbf{1}_{\mathcal{D}_X} & \end{array}$$

Consider the following cartesian diagrams Sc1-Sc3 explained below.

$$\begin{array}{ccc} \begin{array}{ccccc} X & & & & \\ \chi_3 \downarrow & \text{---} S_3 \text{---} & & & \\ X^2 & \xrightarrow{\sigma_3} & X & & \\ \chi_1 \downarrow & & \chi_2 \downarrow & \text{---} S_2 \text{---} & \\ X & \xrightarrow{\sigma_1} & Y & \xrightarrow{\sigma_2} & X \end{array} & \begin{array}{ccccc} X & & & & \\ \Delta \downarrow & & & & \\ X^2 & \xrightarrow{P} & X & & \\ \chi_1 \downarrow & & \chi_2 \downarrow & \text{---} S_2 \text{---} & \\ X & \xrightarrow{G} & Y & \xrightarrow{\sigma_2} & X \end{array} & \begin{array}{ccccc} X & & & & \\ \chi_3 \downarrow & \text{---} S_3 \text{---} & & & \\ X^2 & \xrightarrow{\sigma_3} & X & & \\ Q \downarrow & & \downarrow G & & \\ X & \xrightarrow{\sigma_1} & Y & & \end{array} \\ \text{Sc1} & \text{Sc2} & \text{Sc3} \end{array}$$

We begin by choosing Sc1 to be a staircase based on $\sigma_1 \star \sigma_2$. Thus S_2 (resp. S_3) is the sequence of steps induced by σ_2 (resp. σ_3) and $S_3 \star S_2$, is the sequence of steps induced by $\sigma_1 \star \sigma_2$. The other two diagrams, Sc2 and Sc3, are obtained by collapsing certain portions of Sc1: In Sc2 we set $\Delta := (|\chi_3|, \mathbf{P})$ and $P := (|\sigma_3|, \mathbf{P})$ while in Sc3 we set $Q := (|\chi_1|, \mathbf{P})$; these labels make sense. Indeed, by assumption $|\sigma_1| \in \mathbf{P}$, so that by base change, $|\sigma_3| \in \mathbf{P}$ and by definition of a staircase, $|\chi_1| \in \mathbf{P}$. Also note that by definition of a staircase, $|\chi_2| = |\sigma_1|$ so that $G = (|\chi_2|, \mathbf{P})$.

It follows that the squares occurring in Sc2 and Sc3 are cartesian and the triangles in them represent staircases based on σ_2 and σ_3 respectively.

In order to prove 4.2.6, let us fix a cartesian diagram \mathcal{S} for the purpose of defining $\Psi_{G, \sigma_1}^{\mathcal{S}}$. Given \mathcal{S} , could we have chosen Sc1 above in such a manner that the resulting square in Sc3 is precisely \mathcal{S} ? The answer is yes; see 4.4.2 below. We shall therefore assume that Sc1 has been chosen so that Sc1 and \mathcal{S} are compatible in the said manner.

For $i = 2, 3$, let $\alpha_i: \chi_i^\boxtimes \sigma_i^\boxtimes \xrightarrow{\sim} S_i^\boxtimes \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$ be the obvious natural isomorphisms (see last two maps in the composition defining (3.4.1.5)). Consider the following diagram of isomorphisms where reduced notation has been employed for

naming the maps (§1.3(ii)).

$$\begin{array}{ccccc}
\boxed{\sigma_1^\boxtimes \sigma_2^\boxtimes} & \xlongequal{\quad} & \sigma_1^\boxtimes \sigma_2^\boxtimes & & \\
\downarrow \phi_{\chi_3 * \chi_1} & & \downarrow \dagger_1 & & \downarrow \phi_{\chi_3 * Q} \\
\chi_3^\boxtimes \chi_1^\boxtimes \sigma_1^\boxtimes \sigma_2^\boxtimes & \xleftarrow{\Psi_{\chi_1, Q}} & \chi_3^\boxtimes Q^\boxtimes \sigma_1^\boxtimes \sigma_2^\boxtimes & & \\
\downarrow \beta_{\sigma_1, \chi_2} & & \downarrow \dagger_2 & & \downarrow \beta_{\sigma_1, G} \\
\chi_3^\boxtimes \sigma_3^\boxtimes \chi_2^\boxtimes \sigma_2^\boxtimes & \xleftarrow{\Psi_{\chi_2, G}} & \chi_3^\boxtimes \sigma_3^\boxtimes G^\boxtimes \sigma_2^\boxtimes & & \\
\downarrow \alpha_3 & & \downarrow \alpha_3 & & \\
S_3^\boxtimes & \xleftarrow{\alpha_2} & S_3^\boxtimes \chi_2^\boxtimes \sigma_2^\boxtimes & \xleftarrow{\Psi_{\chi_2, G}} & S_3^\boxtimes G^\boxtimes \sigma_2^\boxtimes \\
\downarrow \phi_{S_3} & & \downarrow \phi_{S_3} & & \downarrow \phi_{S_3} \\
\boxed{1_{\mathcal{D}_X}} & \xleftarrow{\alpha_2} & \chi_2^\boxtimes \sigma_2^\boxtimes & \xleftarrow{\Psi_{\chi_2, G}} & \boxed{G^\boxtimes \sigma_2^\boxtimes} \\
\uparrow \phi_{\Delta * P} & & \uparrow \phi_{\Delta * P} & & \downarrow \phi_{\Delta * \chi_1}^{-1} \\
\Delta^\boxtimes P^\boxtimes & \xleftarrow{\alpha_2} & \Delta^\boxtimes P^\boxtimes \chi_2^\boxtimes \sigma_2^\boxtimes & \xleftarrow{\beta_{G, \chi_2}} & \Delta^\boxtimes \chi_1^\boxtimes G^\boxtimes \sigma_2^\boxtimes
\end{array}$$

The unnamed rectangles commute by functoriality. In \dagger_1 , upon canceling off the common factor $\sigma_1^\boxtimes \sigma_2^\boxtimes$ from each vertex, we are left with objects in which every sequence is composed only of maps labeled P . Therefore \dagger_1 commutes by pseudofunctoriality. Similarly, for \dagger_3 , we first cancel off σ_2^\boxtimes from each vertex and then pseudofunctoriality of $(-)^{\times}$ applies. Finally, for \dagger_2 , we first cancel χ_3^\boxtimes on the left and σ_2^\boxtimes on the right and then use transitivity of β . Thus the preceding diagram commutes. Now Lemma 4.2.6 follows by looking at the outer border comprising the framed objects and the composite maps between them.

We used the following lemma in the above proof.

LEMMA 4.4.2. *Let \mathcal{S} be a cartesian square of labeled sequences as follows where G and Q have length one.*

$$\begin{array}{ccc}
X' & \xrightarrow{\sigma_3} & Y' \\
Q \downarrow & & \downarrow G \\
X & \xrightarrow{\sigma_1} & Y
\end{array}$$

Let $\sigma_2: Y' \Rightarrow Y$ be a sequence such that $|\sigma_2| = |G|$. Then there exists a cartesian square \mathcal{T} as follows

$$\begin{array}{ccc}
X' & \xrightarrow{\sigma_3} & Y' \\
\sigma_4 \Downarrow & & \Downarrow \sigma_2 \\
X & \xrightarrow{\sigma_1} & Y
\end{array}$$

such that $|\sigma_4| = |Q|$.

PROOF. If σ_2 has length one then the result holds trivially with $\mathcal{T} = \mathcal{S}$. Otherwise, we may decompose σ_2 as $Y' \xrightarrow{H} Y_1 \xrightarrow{\rho} Y$. We then proceed by induction on

the length of σ_1 . Suppose σ_1 has length more than two. Then we may decompose \mathcal{S} as shown on the left below.

$$\begin{array}{ccccc}
 X' & \rightrightarrows & W' & \xrightarrow{E'} & Z' & \xrightarrow{F'} & Y' \\
 \downarrow & & \downarrow & & \downarrow K & & \downarrow G \\
 X & \rightrightarrows & W & \xrightarrow{E} & Z & \xrightarrow{F} & Y
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \tilde{X} & \rightrightarrows & \tilde{W} & \xrightarrow{\tilde{E}} & \tilde{Z} & \xrightarrow{\tilde{F}} & Y' \\
 \downarrow & & \downarrow & & \downarrow \tilde{K} & & \downarrow H \\
 X_1 & \rightrightarrows & W_1 & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\
 \Downarrow & & \Downarrow & & \Downarrow \kappa & & \Downarrow \rho \\
 X & \rightrightarrows & W & \xrightarrow{E} & Z & \xrightarrow{F} & Y
 \end{array}$$

Let us choose any cartesian diagram \mathcal{T}_0 as shown on the right above. Then Z' and \tilde{Z} are universal for the same fibered diagram, and so there are canonical inverse isomorphisms $i: \tilde{Z} \xrightarrow{\sim} Z'$ and $j: Z' \rightarrow \tilde{Z}$ such that $|\tilde{F}|j = |F'|$ and $|\kappa||\tilde{K}|j = |K|$. Let \mathcal{T}_1 be the diagram obtained by making the following replacements in \mathcal{T}_0

$$\tilde{Z} \rightsquigarrow Z, \quad \tilde{F} \rightsquigarrow F', \quad \tilde{K} \rightsquigarrow K, \quad \tilde{E} \rightsquigarrow (i|E|, \lambda),$$

where λ is the label of E . Then \mathcal{T}_1 is also a cartesian diagram and it agrees with \mathcal{S} in the rightmost column of squares. We proceed in a similar manner replacing one by one the vertices and edges in the top row of \mathcal{T}_i with that of \mathcal{S} . The case when σ_1 has length one or two is handled using similar (and simpler) arguments. \square

4.5. Proof of Lemma 4.2.7. This is the third and last of the three lemmas used in the proof of Theorem 4.2.1. Its proof is somewhat longer. The compatibility of base-change isomorphisms with the fundamental isomorphism plays a role here.

4.5.1. Let us begin by recalling what has to be proven. The cartesian diagram under consideration is

$$(4.5.1.1) \quad \begin{array}{ccccc}
 X & \xrightarrow{\sigma_1} & A & \xrightarrow{\sigma_2} & B \\
 & & \sigma_3 \downarrow & & \downarrow \sigma_4 \\
 & & C & \xrightarrow{\sigma_5} & D \xrightarrow{\sigma_6} X
 \end{array}$$

where the induced composite map $X \rightarrow X$ is the identity. Then the assertion is that the following diagram of isomorphisms commutes where the Φ 's have the obvious subscripts.

$$(4.5.1.2) \quad \begin{array}{ccc}
 \sigma_1^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes \sigma_6^\boxtimes & \xrightarrow{\text{via } \beta_{\sigma_5, \sigma_4}} & \sigma_1^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes \sigma_6^\boxtimes \\
 \searrow \Phi & & \swarrow \Phi \\
 & \mathbf{1}_{\mathcal{D}_X} &
 \end{array}$$

A quick remark on conventions: in the above context, we shall also allow σ_6 to be the empty sequence, and in such a case we declare $D = X$ and $\sigma_6^\boxtimes = \mathbf{1}_{\mathcal{D}_X}$.

Our immediate goal is to show that the following restrictions may be imposed on the hypothesis of Lemma 4.2.7 (or (4.5.1.1) above) without loss of generality.

- (i) The sequences $\sigma_2, \sigma_3, \sigma_4$ and σ_5 , all have length one.
- (ii) The tail sequence σ_6 is empty.
- (iii) The head sequence σ_1 is a single map with label P.

4.5.2. *Reduction to (i)*: We prove this by induction on the sum of the lengths of σ_4 and σ_5 .

Suppose $\sigma_5 = \mu_1 \star \mu_2$ and let

$$\begin{array}{ccccccc} X & \xrightarrow{\sigma_1} & A & \xrightarrow{\nu_1} & B_1 & \xrightarrow{\nu_2} & B \\ & & \sigma_3 \downarrow & & \theta \downarrow & & \downarrow \sigma_4 \\ & & C & \xrightarrow{\mu_1} & D_1 & \xrightarrow{\mu_2} & D \xrightarrow{\sigma_6} X \end{array}$$

be the corresponding induced cartesian diagram. Consider the following diagram of isomorphisms

$$\begin{array}{ccccc} \sigma_1^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes \sigma_6^\boxtimes & \xrightarrow{\text{via } \beta_{\sigma_5, \sigma_4}} & \sigma_1^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes \sigma_6^\boxtimes & & \\ & \searrow \Phi & \uparrow \dagger & \swarrow \Phi & \\ & & 1_{\mathcal{D}_X} & & \\ & \swarrow \Phi & \uparrow \dagger_1 & \searrow \Phi & \\ \sigma_1^\boxtimes \sigma_3^\boxtimes \mu_1^\boxtimes \mu_2^\boxtimes \sigma_6^\boxtimes & \xrightarrow[\beta_{\mu_1, \theta}]{\text{via}} & \sigma_1^\boxtimes \nu_1^\boxtimes \theta^\boxtimes \mu_2^\boxtimes \sigma_6^\boxtimes & \xrightarrow[\beta_{\mu_2, \sigma_4}]{\text{via}} & \sigma_1^\boxtimes \nu_1^\boxtimes \nu_2^\boxtimes \sigma_4^\boxtimes \sigma_6^\boxtimes \end{array}$$

where the Φ 's have the obvious subscripts. The outer border of the above diagram commutes by definition of $\beta_{\sigma_5, \sigma_4}$. Therefore, if by induction hypothesis, we assume that \dagger_1, \dagger_2 commute, then so does \dagger .

A similar argument works if we begin with a decomposition of σ_4 into two sequences. Thus by induction, we obtain reduction to (i).

4.5.3. *Reduction to (ii)*: We may assume that the (i) has been applied. We proceed in two stages.

Set $g := |\sigma_1 \star \sigma_3 \star \sigma_5|$ and $G := (g, P)$. Consider the following diagram

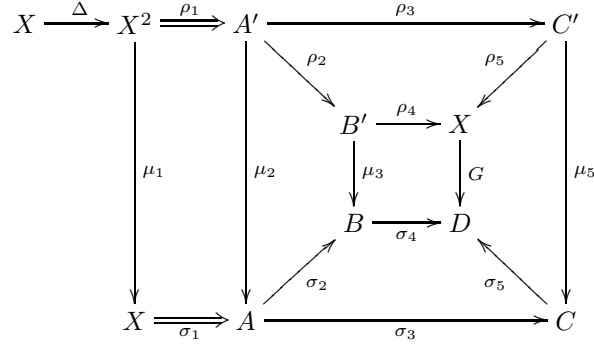
$$(4.5.3.1) \quad \begin{array}{ccc} \sigma_1^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes & \xrightarrow{\text{via } \beta_{\sigma_5, \sigma_4}} & \sigma_1^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes \\ & \searrow \Psi_3 & \swarrow \Psi_2 \\ & & G^\boxtimes \end{array}$$

where $\Psi_3 := \Psi_{G, \sigma_1 \star \sigma_3 \star \sigma_5}^{\mathcal{S}}$ and $\Psi_2 := \Psi_{G, \sigma_1 \star \sigma_2 \star \sigma_4}^{\mathcal{T}}$ for some suitable choice of \mathcal{S} and \mathcal{T} . We claim that to prove that (4.5.1.2) commutes it suffices to find a choice for \mathcal{S} and \mathcal{T} such that (4.5.3.1) commutes. The claim follows by looking at the following diagram

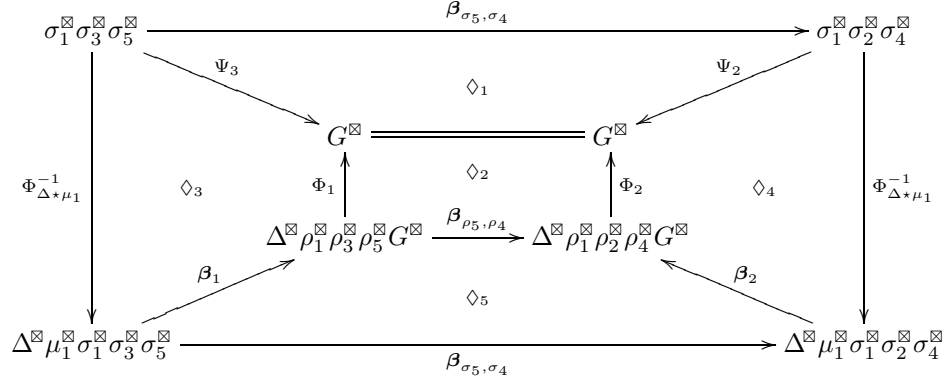
$$\begin{array}{ccc} \sigma_1^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes \sigma_6^\boxtimes & \xrightarrow{\text{via } \beta_{\sigma_5, \sigma_4}} & \sigma_1^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes \sigma_6^\boxtimes \\ & \searrow \Psi_3(\sigma_6^\boxtimes) & \swarrow \Psi_2(\sigma_6^\boxtimes) \\ & & G^\boxtimes \sigma_6^\boxtimes \\ & \searrow \Phi & \swarrow \Phi \\ & & F^\boxtimes \end{array}$$

where the outer border is (4.5.1.2), the lower two sub-triangles on either side commute by 4.2.6, and commutativity of the subtriangle on the top reduces to that of (4.5.3.1).

Now consider a cartesian diagram as follows, which is obtained by taking a product of (4.5.1.1) (with a truncated tail) with G .



Let Ψ_3 and Ψ_2 (and in particular, \mathcal{S} and \mathcal{T}) of (4.5.3.1) be defined via the cartesian squares in the preceding diagram. Now consider the following diagram of isomorphisms.



The morphisms are indicated in reduced notation using the following.

$$\Phi_1 = \Phi_{\Delta \star \rho_1 \star \rho_3 \star \rho_5} \quad \Phi_2 = \Phi_{\Delta \star \rho_1 \star \rho_2 \star \rho_4} \quad \beta_1 = \beta_{\sigma_1 \star \sigma_3 \star \sigma_5, G} \quad \beta_2 = \beta_{\sigma_1 \star \sigma_2 \star \sigma_4, G}$$

The outer border of the preceding diagram commutes for functorial reasons. From the definition of Ψ_3 and Ψ_2 we see that \diamond_3 and \diamond_4 commute (cf. 4.1.1). In \diamond_5 , if we first cancel off Δ^\boxtimes on the right, then the transpose of the resulting diagram may be expanded as follows.

$$\begin{array}{ccccc} \mu_1^\boxtimes \sigma_1^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes & \xrightarrow{\beta_{\sigma_1, \mu_2}} & \rho_1^\boxtimes \mu_2^\boxtimes \sigma_3^\boxtimes \sigma_5^\boxtimes & \xrightarrow{\beta_{\sigma_3, \mu_5}} & \rho_1^\boxtimes \rho_3^\boxtimes \mu_5^\boxtimes \sigma_5^\boxtimes & \xrightarrow{\beta_{\sigma_5, G}} & \rho_1^\boxtimes \rho_3^\boxtimes \rho_5^\boxtimes G^\boxtimes \\ \beta_{\sigma_5, \sigma_4} \downarrow & & \downarrow \beta_{\sigma_5, \sigma_4} & & \downarrow \beta_{\rho_5, \rho_4} & & \\ \mu_1^\boxtimes \sigma_1^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes & \xrightarrow{\beta_{\sigma_1, \mu_2}} & \rho_1^\boxtimes \mu_2^\boxtimes \sigma_2^\boxtimes \sigma_4^\boxtimes & \xrightarrow{\beta_{\sigma_2, \mu_3}} & \rho_1^\boxtimes \rho_2^\boxtimes \mu_3^\boxtimes \sigma_4^\boxtimes & \xrightarrow{\beta_{\sigma_4, G}} & \rho_1^\boxtimes \rho_2^\boxtimes \rho_4^\boxtimes G^\boxtimes \end{array}$$

Once again, reduced notation has been applied here. The rectangle on the left commutes for functorial reasons while the one on the right commutes by the cube lemma. Thus \diamond_5 commutes.

It follows that \diamond_1 commutes if and only if \diamond_2 commutes. Now consider the following two diagrams.

(4.5.3.2)

$$\begin{array}{ccc}
 X \xrightarrow{\Delta \star \rho_1} A' \xrightarrow{\rho_2} B' & & \Delta^{\boxtimes} \rho_1^{\boxtimes} \rho_3^{\boxtimes} \rho_5^{\boxtimes} \xrightarrow{\text{via } \beta_{\rho_5, \rho_4}} \Delta^{\boxtimes} \rho_1^{\boxtimes} \rho_2^{\boxtimes} \rho_4^{\boxtimes} \\
 \rho_3 \downarrow & & \searrow \Phi \quad \swarrow \Phi \\
 C' \xrightarrow{\rho_5} D' = X & & \mathbf{1}_{\mathcal{D}_X}
 \end{array}$$

The one on the left is an instance of the cartesian diagram in (4.5.1.1) with tail sequence empty. The one on the right corresponds to (4.5.1.2). Now \diamond_2 , upon the canceling of G^{\boxtimes} from each vertex, results in the diagram shown on the right in (4.5.3.2). Therefore, proving that \diamond_1 commutes, reduces to proving that the diagram on the right in (4.5.3.2) commutes.

In summary, if the diagram on the right in (4.5.3.2) commutes then so does \diamond_1 which is nothing but (4.5.3.1), and hence so does (4.5.1.2). We have therefore achieved reduction to (ii).

4.5.4. *Reduction to (iii)*: We may now assume that the restrictions of (i) and (ii) apply.

Let us set $H := (|\sigma_1|, P)$. Now consider the following diagram of isomorphisms where reduced notation has been used for the morphisms, the Φ 's have the obvious subscripts and \mathbb{S} is a suitable fixed diagram used in defining $\Psi_{H, \sigma_1}^{\mathbb{S}}$.

$$\begin{array}{ccc}
 \sigma_1^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} & \xrightarrow{\beta_{\sigma_5, \sigma_4}} & \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} \\
 \Psi_{H, \sigma_1}^{\mathbb{S}} \downarrow & \searrow \Phi \quad \swarrow \Phi & \downarrow \Psi_{H, \sigma_1}^{\mathbb{S}} \\
 & \mathbf{1}_{\mathcal{D}_X} & \\
 H^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} & \xrightarrow{\beta_{\sigma_5, \sigma_4}} & H^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes}
 \end{array}$$

The outer border commutes for functorial reasons. The two triangles on the left and right sides commute by 4.2.6. Therefore for the top triangle to commute, it suffices that the bottom one commutes. This is precisely reduction to (iii).

4.5.5. Once all the reductions (i)-(iii) have been carried out, we may replace the diagrams of (4.5.1.1) and (4.5.1.2) by the following ones where each σ_i has length one, σ_1 has label P and $|\sigma_1 \star \sigma_3 \star \sigma_5| = 1_X$.

$$\begin{array}{ccc}
 X \xrightarrow{\sigma_1} A \xrightarrow{\sigma_2} B & & \sigma_1^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} \xrightarrow{\beta_{\sigma_5, \sigma_4}} \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} \\
 \sigma_3 \downarrow & & \searrow \phi \quad \swarrow \phi \\
 C \xrightarrow{\sigma_5} X & & \mathbf{1}_{\mathcal{D}_X}
 \end{array}$$

Set $t_4 := |\sigma_1 \star \sigma_2|: X \rightarrow B$ and $t_5 := |\sigma_1 \star \sigma_3|: X \rightarrow C$, so that t_4, t_5 are sections of $|\sigma_4|, |\sigma_5|$ respectively. Therefore $t_4, t_5 \in P$. For $i = 4, 5$ set $\theta_i := (t_i, P)$. Now taking a fibered product of the sequence $X \xrightarrow{\theta_4} B \xrightarrow{\sigma_4} X$ with $X \xrightarrow{\theta_5} C \xrightarrow{\sigma_5} X$ results in a

diagram as follows for suitably determined θ_2 and θ_3 .

$$(4.5.5.2) \quad \begin{array}{ccccc} X & \xrightarrow{\theta_5} & C & \xrightarrow{\sigma_5} & X \\ \downarrow \theta_4 & & \downarrow \theta_3 & & \downarrow \theta_4 \\ B & \xrightarrow{\theta_2} & A & \xrightarrow{\sigma_2} & B \\ \downarrow \sigma_4 & & \downarrow \sigma_3 & & \downarrow \sigma_4 \\ X & \xrightarrow{\theta_5} & C & \xrightarrow{\sigma_5} & X \end{array}$$

Set $t := |\theta_4 \star \theta_2| = |\theta_5 \star \theta_3|: X \rightarrow A$. We claim that $t = |\sigma_1|$. Indeed, note that for $i = 2, 3$, we have $|\sigma_i|t = |\sigma_i||\sigma_1|$ because

$$|\sigma_2|t = |\sigma_2||\theta_2||\theta_4| = |\theta_4| := |\sigma_2||\sigma_1|, \quad |\sigma_3|t = |\sigma_3||\theta_3||\theta_5| = |\theta_5| := |\sigma_3||\sigma_1|.$$

Therefore, by the universal property of fibered products for the cartesian square in (4.5.5.1), we conclude that $t = |\sigma_1|$. In particular, since σ_1 and θ_i for $i = 2, 3, 4, 5$ have label P, by pseudofunctoriality there result natural isomorphisms

$$(4.5.5.3) \quad \sigma_1^\boxtimes \xrightarrow{\sim} \theta_4^\boxtimes \theta_2^\boxtimes \xrightarrow{\sim} \theta_5^\boxtimes \theta_3^\boxtimes.$$

Now consider the following diagram of isomorphisms. The superscript \boxtimes has been omitted from each term and reduced notation has been used for the morphisms. The morphisms denoted by π_i are obtained using (4.5.5.3). The unlabeled ones are obtained using the suitably determined fundamental isomorphism.

$$(4.5.5.4) \quad \begin{array}{ccccc} \sigma_1 \sigma_3 \sigma_5 & \xrightarrow{\beta_{\sigma_5, \sigma_4}} & \sigma_1 \sigma_2 \sigma_4 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \theta_4 \theta_2 \sigma_3 \sigma_5 & \xrightarrow{\beta_{\sigma_5, \sigma_4}} & \theta_4 \theta_2 \sigma_2 \sigma_4 \\ \downarrow \pi_3 & \searrow \beta_{\sigma_3, \theta_5} & \downarrow \pi_4 \\ & \theta_4 \sigma_4 \theta_5 \sigma_5 & \theta_4 \sigma_4 \\ & \downarrow & \downarrow \\ & \theta_5 \sigma_5 & \theta_5 \sigma_5 \theta_4 \sigma_4 \\ & \swarrow & \searrow \beta_{\theta_4, \sigma_2} \\ \theta_5 \theta_3 \sigma_3 \sigma_5 & \xrightarrow{\beta_{\sigma_5, \sigma_4}} & \theta_5 \theta_3 \sigma_2 \sigma_4 \end{array}$$

\dagger_1 (between $\theta_4 \theta_2 \sigma_3 \sigma_5$ and $\theta_4 \theta_2 \sigma_2 \sigma_4$)
 \dagger_2 (between $\theta_4 \theta_2 \sigma_3 \sigma_5$ and $\theta_5 \theta_3 \sigma_3 \sigma_5$)
 \dagger_3 (between $\theta_4 \theta_2 \sigma_2 \sigma_4$ and $\theta_5 \theta_3 \sigma_2 \sigma_4$)
 \dagger_4 (between $\theta_5 \sigma_5$ and $\theta_5 \sigma_5 \theta_4 \sigma_4$)

Let us verify that this diagram commutes. The unmarked subdiagrams commute for functorial reasons and so only \dagger_i need to be considered. For these we use the compatibility of the fundamental isomorphism with base change (3.2.5(ii)). For instance, in \dagger_1 , we first cancel the common term θ_4^\boxtimes and then use 3.2.5(ii) for the

following subdiagram of (4.5.5.2).

$$\begin{array}{ccccc} B & \xrightarrow{\theta_2} & A & \xrightarrow{\sigma_2} & B \\ \downarrow \sigma_4 & & \downarrow \sigma_3 & & \downarrow \sigma_4 \\ X & \xrightarrow{\theta_5} & C & \xrightarrow{\sigma_5} & X \end{array}$$

A similar argument works for \dagger_2 , \dagger_3 and \dagger_4 .

Since (4.5.5.4) commutes, to finish the proof of commutativity of the diagram on right in (4.5.5.1), it suffices to show that the following composites obtained from (4.5.5.4) give $\Phi_{\sigma_1 \star \sigma_3 \star \sigma_5}$ and $\Phi_{\sigma_1 \star \sigma_2 \star \sigma_4}$ respectively.

$$(4.5.5.5) \quad \sigma_1^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} \xrightarrow{\pi_1} \theta_4^{\boxtimes} \theta_2^{\boxtimes} \sigma_3^{\boxtimes} \sigma_5^{\boxtimes} \xrightarrow[\beta_{\sigma_3, \theta_5}]{\text{via}} \theta_4^{\boxtimes} \sigma_4^{\boxtimes} \theta_5^{\boxtimes} \sigma_5^{\boxtimes} \xrightarrow{\Phi} \theta_5^{\boxtimes} \sigma_5^{\boxtimes} \xrightarrow{\Phi} \mathbf{1}_{\mathcal{D}_X}$$

$$(4.5.5.6) \quad \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} \xrightarrow{\pi_4 \pi_2} \theta_5^{\boxtimes} \theta_3^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} \xrightarrow[\beta_{\sigma_2, \theta_4}]{\text{via}} \theta_5^{\boxtimes} \sigma_5^{\boxtimes} \theta_4^{\boxtimes} \sigma_4^{\boxtimes} \xrightarrow{\Phi} \theta_4^{\boxtimes} \sigma_4^{\boxtimes} \xrightarrow{\Phi} \mathbf{1}_{\mathcal{D}_X}$$

We only consider the second case, viz., that of showing $\Phi_{\sigma_1 \star \sigma_2 \star \sigma_4} = (4.5.5.6)$. The first one is proved in a similar manner.

Consider the following cartesian diagram where κ_i for $i = 2, 3, 4, 5$ is constructed via the cartesian condition and κ_1 is the diagonal map with label P.

$$\begin{array}{ccccc} X & & & & \\ \downarrow \kappa_1 & & & & \\ \bullet & \xrightarrow{\kappa_2} & X & & \\ \downarrow \kappa_3 & & \downarrow \theta_5 & & \\ \bullet & \xrightarrow{\kappa_4} & C & \xrightarrow{\sigma_5} & X \\ \downarrow \kappa_5 & & \downarrow \theta_3 & & \downarrow \theta_4 \\ X & \xrightarrow{\sigma_1} & A & \xrightarrow{\sigma_2} & B & \xrightarrow{\sigma_4} & X \end{array}$$

By definition, $|\theta_4| := |\sigma_1 \star \sigma_2|$ and as verified above, $|\theta_5 \star \theta_3| = |\sigma_1|$ and $|\theta_5 \star \sigma_5| = 1_X$. Therefore, the preceding diagram is a staircase based on $\sigma_1 \star \sigma_2 \star \sigma_4$.

Now consider the following diagram of isomorphisms.

$$\begin{array}{ccccccc} \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \theta_5^{\boxtimes} \theta_3^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \theta_5^{\boxtimes} \sigma_5^{\boxtimes} \theta_4^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \mathbf{1}_{\mathcal{D}_X} \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ \kappa_1^{\boxtimes} \kappa_3^{\boxtimes} \kappa_5^{\boxtimes} \sigma_1^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \kappa_1^{\boxtimes} \kappa_2^{\boxtimes} \theta_5^{\boxtimes} \theta_3^{\boxtimes} \sigma_2^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \kappa_1^{\boxtimes} \kappa_2^{\boxtimes} \theta_5^{\boxtimes} \sigma_5^{\boxtimes} \theta_4^{\boxtimes} \sigma_4^{\boxtimes} & \longrightarrow & \mathbf{1}_{\mathcal{D}_X} \end{array}$$

Here the top row gives (4.5.5.6) and the rest of the outer border spells out the definition of $\Phi_{\sigma_1 \star \sigma_2 \star \sigma_4}$. Each vertical map is induced in the obvious way by a suitably determined fundamental isomorphism. The rectangle on the left, upon the cancellation of the common term $\sigma_2^{\boxtimes} \sigma_4^{\boxtimes}$, consists only of P-labeled maps and hence commutes by pseudofunctoriality. The other two rectangles commute for functorial reasons. Thus the preceding diagram commutes. In particular, $\Phi_{\sigma_1 \star \sigma_2 \star \sigma_4} = (4.5.5.6)$.

Thus we have shown that the diagram on the right in (4.5.5.1) commutes. This completes the proof of Lemma 4.2.7.

4.6. Canonicity of $\Psi_{-, -}$. Now that the proof of Theorem 4.2.1 has been completed, we discuss some of the immediate consequences.

PROPOSITION 4.6.1. *The isomorphism $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ of (4.1.1.1) is independent of the choice of \mathcal{S} .*

PROOF. First we claim that for any sequence σ and any available choice of diagram \mathcal{T} , the isomorphism $\Psi_{\sigma, \sigma}^{\mathcal{T}}$ is the identity. Indeed, the cocycle condition says that

$$\Psi_{\sigma, \sigma}^{\mathcal{T}} \Psi_{\sigma, \sigma}^{\mathcal{T}} = \Psi_{\sigma, \sigma}^{\mathcal{T}}$$

and since $\Psi_{\sigma, \sigma}^{\mathcal{T}}$ is an isomorphism, it is necessarily the identity.

Now suppose \mathcal{S} and \mathcal{S}' are two diagrams through which $\Psi_{\sigma_1, \sigma_2}^-$ is defined. Let \mathcal{T} be any diagram through which $\Psi_{\sigma_2, \sigma_2}^-$ is defined. The cocycle condition implies that the following holds

$$\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}} \Psi_{\sigma_2, \sigma_2}^{\mathcal{T}} = \Psi_{\sigma_1, \sigma_2}^{\mathcal{S}'}$$

and since $\Psi_{\sigma_2, \sigma_2}^{\mathcal{T}}$ is the identity, therefore $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}} = \Psi_{\sigma_1, \sigma_2}^{\mathcal{S}'}$. \square

DEFINITION 4.6.2. For any two sequences σ_1, σ_2 such that $|\sigma_1| = |\sigma_2|$ we define $\Psi_{\sigma_1, \sigma_2}$ to be $\Psi_{\sigma_1, \sigma_2}^{\mathcal{S}}$ of (4.1.1.1) for any available \mathcal{S} .

PROPOSITION 4.6.3. *The isomorphism $\Psi_{-, -}$ is reflexive and symmetric, i.e., the following hold.*

- (i) $\Psi_{\sigma, \sigma}$ is the identity for any σ .
- (ii) $\Psi_{\sigma_1, \sigma_2} = \Psi_{\sigma_2, \sigma_1}^{-1}$ for any σ_1, σ_2 such that $|\sigma_1| = |\sigma_2|$.

PROOF. This is an immediate consequence of the cocycle rule. See proof of 4.6.1. \square

5. Proofs III (old isomorphisms and linearity)

There are two objectives in this section. The first is to show that the isomorphism $\Psi_{-, -}$ defined in the previous section recovers the isomorphisms of the input data in §2.1 and more generally their labeled counterparts. This is achieved in §5.1 and §5.2. The other objective, discussed in §5.3, is of proving a linearity rule satisfied by $\Psi_{-, -}$.

5.1. Recovering $\mathbf{C}_{-, -}$ and $\Phi_{-, -}$. Here record that the isomorphisms $\mathbf{C}_{-, -}$ and $\Phi_{-, -}$ are expressible in terms of $\Psi_{-, -}$.

Before we begin with the case of $\mathbf{C}_{-, -}$, we need a lemma.

LEMMA 5.1.1. *Let σ_1 be the sequence $X \xrightarrow{G_1} Y \xrightarrow{G_2} Z \xrightarrow{G_3} X$ and σ_2 the sequence $X \xrightarrow{G_1} Y \xrightarrow{G_4} X$ such that G_2, G_3, G_4 have label \mathbf{O} , $|G_3||G_2| = |G_4|$ and $|\sigma_i| = 1_X$. Then the following diagram commutes*

$$\begin{array}{ccc} G_1^{\boxtimes} G_2^{\boxtimes} G_3^{\boxtimes} & \xrightarrow{\text{via } \mathbf{C}_{G_2, G_3}} & G_1^{\boxtimes} G_4^{\boxtimes} \\ & \searrow \Phi_{\sigma_1} \quad \swarrow \Phi_{\sigma_2} & \\ & \mathbf{1}_{\mathcal{D}_X} & \end{array}$$

$$\begin{array}{ccccc}
X & & & & \\
\alpha_1 \downarrow & & & & \\
X'' & \xrightarrow{\alpha_2} & X & & \\
\alpha_{37} \downarrow & \text{\texttt{5}} & \downarrow \alpha_{48} & & \\
X & \xrightarrow{G_1} & Y & \xrightarrow{G_4} & X
\end{array}$$
$$\begin{array}{ccc} a_4^\times a_8^\times g_2^\square g_3^\square & \longrightarrow & a_4^\times a_6^\square a_9^\times g_3^\square \\ \downarrow & & \downarrow \\ a_{48}^\times g_4^\square & \longrightarrow & \mathbf{1}_{\mathcal{D}_X} \end{array}$$

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \times_Z X & \xrightarrow{\tilde{F}_1} & Y \times_Z X & \xrightarrow{\tilde{F}_2} & X \\ & & \downarrow F'_3 & & \downarrow & & \downarrow F_3 \\ & & X & \xrightarrow{F_1} & Y & \xrightarrow{F_2} & Z \end{array}$$
$$\begin{array}{ccccccc}
F_1^\boxtimes F_2^\boxtimes & \longrightarrow & \Delta^\boxtimes F_3' \boxtimes F_1^\boxtimes F_2^\boxtimes & \longrightarrow & \Delta^\boxtimes \tilde{F}_1 \boxtimes \tilde{F}_2 \boxtimes F_3^\boxtimes & \longrightarrow & F_3^\boxtimes \\
\downarrow \scriptstyle C_{F_1, F_2} & & \downarrow \scriptstyle C_{F_1, F_2} & & \downarrow \scriptstyle C_{\tilde{F}_1, \tilde{F}_2} & & \parallel \\
F_3^\boxtimes & \longrightarrow & \Delta^\boxtimes F_3' \boxtimes F_3^\boxtimes & \longrightarrow & \Delta^\boxtimes \tilde{F}_3 \boxtimes F_3^\boxtimes & \longrightarrow & F_3^\boxtimes
\end{array}$$

The square on the left commutes for functorial reasons, the one in the middle commutes by transitivity of base-change, and the one on the right commutes by 5.1.1. Thus the diagram commutes. By reflexivity of $\Psi_{-, -}$ (4.6.3(i)) the bottom row is the identity and hence the proposition follows by looking at the outer border of the diagram. \square

Next, we look at the fundamental isomorphism.

LEMMA 5.1.3. (Recovering Φ_-). *Let σ be a sequence, such that $|\sigma|$ is an identity map, say 1_X . Set $I := (1_X, P)$. Then $\Phi_\sigma = \Psi_{I, \sigma}$.*

PROOF. By definition, $\Psi_{I, \sigma}$ is defined via the following diagram where we may assume that intermediate vertical maps in the cartesian square S are all P -labeled identity maps.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X & \xrightarrow{\sigma' = \sigma} & X \\ & & \parallel & \searrow s & \parallel \\ & & X & \xrightarrow{\sigma} & X \end{array}$$

$I' = I$ on the left vertical arrow, I on the right vertical arrow.

In this case the diagonal map is the identity so that $\Delta^\boxtimes = I^\boxtimes = 1_{\mathcal{D}_X}$. By looking at the outer border between the framed vertices below we see that it suffices to prove that the following diagram commutes.

$$\begin{array}{ccccc} \Delta^\boxtimes I'^\boxtimes \sigma^\boxtimes & \xrightarrow{\text{via } \beta_s^{-1}} & \Delta^\boxtimes \sigma'^\boxtimes I^\boxtimes & \xrightarrow{\text{via } \Phi_{\Delta * \sigma'}} & I^\boxtimes \\ \downarrow \text{via } \Phi_{\Delta * I'} & \nearrow \dagger & \downarrow \text{via } \beta_s^{-1} & \nearrow \dagger_1 & \downarrow \\ I'^\boxtimes \sigma^\boxtimes & \xrightarrow{\dagger_2} & \sigma'^\boxtimes I^\boxtimes & \xrightarrow{\text{via } \Phi_{\sigma'}} & I^\boxtimes \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\sigma^\boxtimes} & \xrightarrow{\quad} & \boxed{\sigma'^\boxtimes} & \xrightarrow{\Phi_{\sigma'}} & \boxed{1_{\mathcal{D}_X}} \end{array}$$

Since Δ and I' are P -labeled, \dagger commutes by pseudofunctoriality of $(-)^{\times}$. For \dagger_2 , we use 3.3.6 while \dagger_1 commutes by 3.5.11. The remaining subdiagrams commute for functorial reasons. \square

5.2. Recovering the base-change isomorphisms. Now, we move on to recovering the base-change isomorphism. Again, we need some preliminaries. These are also used in §5.3.

LEMMA 5.2.1. *Let $\sigma_i: X \rightrightarrows X$ for $i = 1, 2$ be sequences such that $|\sigma_i| = 1_X$. Then the following diagram commutes.*

$$\begin{array}{ccc} \sigma_1^\boxtimes \sigma_2^\boxtimes & \xrightarrow{\text{via } \Phi_{\sigma_2}} & \sigma_1^\boxtimes \\ \downarrow \text{via } \Phi_{\sigma_1} & \searrow \Phi_{\sigma_1 * \sigma_2} & \downarrow \Phi_{\sigma_1} \\ \sigma_2^\boxtimes & \xrightarrow{\Phi_{\sigma_2}} & 1_{\mathcal{D}_X} \end{array}$$

PROOF. The outer diagram commutes for functorial reasons. For the lower triangle we use 4.2.6 with G as the identity map on X and then use 5.1.3 followed by 3.5.11. \square

LEMMA 5.2.2. *Consider a cartesian diagram as follows where $|\sigma_1| = 1_X$ and $|\sigma_3| = 1_Z$.*

$$\begin{array}{ccc} Z & \xrightarrow{\sigma_3} & Z \\ \sigma_2 \downarrow & & \downarrow \sigma_2 \\ X & \xrightarrow{\sigma_1} & X \end{array}$$

Then the following diagram commutes.

$$\begin{array}{ccc} \sigma_2^{\boxtimes} \sigma_1^{\boxtimes} & \xrightarrow{\beta_{\sigma_1, \sigma_2}} & \sigma_3^{\boxtimes} \sigma_2^{\boxtimes} \\ & \searrow \text{via } \Phi_{\sigma_1} & \swarrow \text{via } \Phi_{\sigma_3} \\ & \sigma_2^{\boxtimes} & \end{array}$$

PROOF. As a first step, we reduce, by induction on the length of σ_2 , to the case where σ_2 has length one. Let $n > 1$ be an integer and assume that the lemma has been verified when σ_2 has length less than n . Now suppose σ_2 has length n . We can decompose σ_2 as $\sigma_4 \star \sigma_5: Z \xrightarrow{\sigma_4} Z_1 \xrightarrow{\sigma_5} X$. Both, σ_4, σ_5 have length less than n . Consider the following cartesian diagram,

$$\begin{array}{ccc} Z & \xrightarrow{\sigma_3} & Z \\ \sigma_4 \downarrow & & \downarrow \sigma_4 \\ Z_1 & \xrightarrow{\sigma_6} & Z_1 \\ \sigma_5 \downarrow & & \downarrow \sigma_5 \\ X & \xrightarrow{\sigma_1} & X \end{array}$$

corresponding to which, there results the following diagram of isomorphisms.

$$\begin{array}{ccccc} \sigma_4^{\boxtimes} \sigma_5^{\boxtimes} \sigma_1^{\boxtimes} & \xrightarrow{\text{via } \beta_{\sigma_1, \sigma_5}} & \sigma_4^{\boxtimes} \sigma_6^{\boxtimes} \sigma_5^{\boxtimes} & \xrightarrow{\text{via } \beta_{\sigma_6, \sigma_4}} & \sigma_3^{\boxtimes} \sigma_4^{\boxtimes} \sigma_5^{\boxtimes} \\ & \searrow \text{via } \Phi_{\sigma_1} & \downarrow \text{via } \Phi_{\sigma_6} & \swarrow \text{via } \Phi_{\sigma_3} & \\ & & \sigma_4^{\boxtimes} \sigma_5^{\boxtimes} & & \end{array}$$

By the induction hypothesis, the two subtriangles commute and hence from the outer border we get the desired reduction.

Henceforth, we assume that σ_2 has length one. For carrying out the next step we need the following definition.

For any labeled sequence σ , given by, say,

$$X_1 \xrightarrow{(f_1, \lambda_1)} X_2 \xrightarrow{(f_2, \lambda_2)} \dots \xrightarrow{(f_n, \lambda_n)} X_{n+1}$$

we define its O-depth by

$$\text{O-depth}(\sigma) = \begin{cases} 0, & \text{if } \lambda_i = \mathbf{P} \text{ for all } i; \\ n + 1 - \inf \{i \mid \lambda_i \in \mathbf{O}\}, & \text{otherwise.} \end{cases}$$

Returning to the proof of the lemma, we now argue by induction on the O-depth of σ_1 . Let us verify that the lemma holds if σ_1 has O-depth 0. Indeed, first note that in this case, the entire staircase associated to σ_1 consists of P-labeled maps only and hence Φ_{σ_1} is the same as the isomorphism $\sigma_1^{\boxtimes} \xrightarrow{\sim} (1_X)^{\times} = \mathbf{1}_{\mathcal{D}_X}$

resulting from the pseudofunctoriality of $(-)^{\times}$. A similar description holds for Φ_{σ_3} . Therefore, using transitivity of base-change successively over the length of σ_1, σ_3 we reduce to proving the lemma when σ_1, σ_3 are assumed to be identity maps with label P. Now we conclude using 3.2.5(i).

Now let $n > 0$ be an integer, assume that the lemma holds whenever σ_1 has O-depth less than n and assume that σ_1 has O-depth n . Let us decompose σ_1 as $\sigma_4 \star G: X \xrightarrow{\sigma_4} Y \xrightarrow{G} X$; here G is a labeled map. Note that σ_4 has O-depth $n - 1$. Set $F := (|\sigma_4|, P)$. Consider the following cartesian diagram where the previously undefined maps are determined by the cartesian condition.

$$\begin{array}{ccccc}
 Z & \xrightarrow{\sigma_7} & Y' & \xrightarrow{G'} & Z \\
 \sigma_2 \downarrow & \nearrow H' & \downarrow \sigma_2' & \nearrow F' & \downarrow \sigma_2 \\
 Z & \xrightarrow{\Delta'} & X' & \xrightarrow{\sigma_6} & Z \\
 \sigma_2 \downarrow & \nearrow \Delta & \downarrow \sigma_2'' & \nearrow \sigma_2 & \downarrow \sigma_2 \\
 X & \xrightarrow{\Delta} & X^2 & \xrightarrow{\sigma_5} & X \\
 \sigma_2 \downarrow & \nearrow H & \downarrow \sigma_2 & \nearrow F & \downarrow \sigma_2 \\
 X & \xrightarrow{\sigma_4} & Y & \xrightarrow{G} & X
 \end{array}$$

Correspondingly, we have the following diagram of isomorphisms where the common superscript \boxtimes has been omitted from each term.

(5.2.2.1)

$$\begin{array}{ccccccc}
 \boxed{\sigma_2 \sigma_4 G} & \xrightarrow{\Phi_1} & \sigma_2 \Delta H \sigma_4 G & \xrightarrow{\beta_1} & \sigma_2 \Delta \sigma_5 F G & & \\
 \downarrow \beta_4 & \searrow \Phi_2 & \downarrow \beta_2 & \searrow \beta_2 & \downarrow \Phi_5 & & \\
 & & \Delta' H' \sigma_2 \sigma_4 G & \xrightarrow{\beta_3} & \Delta' \sigma_2'' H \sigma_4 G & \xrightarrow{\beta_1} & \Delta' \sigma_2'' \sigma_5 F G \\
 & & \downarrow \beta_4 & & \downarrow \beta_5 & & \downarrow \Phi_3 \\
 \sigma_7 \sigma_2' G & \xrightarrow{\Phi_2} & \Delta' H' \sigma_7 \sigma_2' G & \xrightarrow{\beta_7} & \Delta' \sigma_6 F' \sigma_2' G & \xrightarrow{\beta_8} & \Delta' \sigma_6 \sigma_2 F G \\
 \downarrow \beta_6 & & \downarrow \beta_6 & & \downarrow \beta_6 & & \downarrow \Phi_3 \\
 \boxed{\sigma_7 G' \sigma_2} & \xrightarrow{\Phi_2} & \Delta' H' \sigma_7 G' \sigma_2 & \xrightarrow{\beta_7} & \Delta' \sigma_6 F' G' \sigma_2 & \xrightarrow{\Phi_6} & F' \sigma_2' G \\
 & & & & & & \downarrow \beta_6 \\
 & & & & & & F' G' \sigma_2
 \end{array}$$

The morphisms are denoted in the reduced notation using the following table.

$\Phi_1 = \Phi_{\Delta \star H}^{-1}$	$\Phi_5 = \Phi_{\Delta \star \sigma_5}$	$\beta_1 = \beta_{\sigma_4, F}$	$\beta_5 = \beta_{\sigma_5, \sigma_2}$
$\Phi_2 = \Phi_{\Delta' \star H'}^{-1}$	$\Phi_6 = \Phi_{\Delta' \star \sigma_6}$	$\beta_2 = \beta_{\Delta, \sigma_2'}$	$\beta_6 = \beta_{G, \sigma_2}$
$\Phi_3 = \Phi_{F \star G}$		$\beta_3 = \beta_{\sigma_2, H}$	$\beta_7 = \beta_{\sigma_7, F'}$
$\Phi_4 = \Phi_{F' \star G'}$		$\beta_4 = \beta_{\sigma_4, \sigma_2'}$	$\beta_8 = \beta_{\sigma_2', F}$

Let us verify that (5.2.2.1) commutes. The unnamed rectangles commute for functorial reasons. By 3.2.5(ii), we deduce commutativity of \blacksquare_4 and \blacksquare_1 , where in

the latter case we first cancel the common factor $\sigma_4^\boxtimes G^\boxtimes$ on the right from each vertex. In \blacksquare_2 , we cancel Δ^\boxtimes on the left and G^\boxtimes on the right and then use the cube lemma. Finally, for \blacksquare_3 , we first cancel the common factor $F^\boxtimes G^\boxtimes$ on the right. Now note that $\Delta \star \sigma_5$ has O-depth $n - 1$. Therefore, by the induction hypothesis, commutativity follows.

To conclude the lemma consider the outer border of (5.2.2.1) redrawn, with restored superscripts, as follows.

$$\begin{array}{ccccc}
\boxed{\sigma_2^\boxtimes \sigma_4^\boxtimes G^\boxtimes} & \xrightarrow[\Phi_1]{\text{via}} & \sigma_2^\boxtimes \Delta^\boxtimes H^\boxtimes \sigma_4^\boxtimes G^\boxtimes & \xrightarrow[\beta_1]{\text{via}} & \sigma_2^\boxtimes \Delta^\boxtimes \sigma_5^\boxtimes F^\boxtimes G^\boxtimes & \xrightarrow[\Phi_5]{\text{via}} & \sigma_2^\boxtimes F^\boxtimes G^\boxtimes \\
\downarrow \text{via } \beta_4 & & & & & & \downarrow \text{via } \Phi_3 \\
\sigma_7^\boxtimes \sigma_2^\boxtimes G^\boxtimes & & & & & & \boxed{\sigma_2^\boxtimes} \\
\downarrow \text{via } \beta_6 & & & & & & \uparrow \text{via } \Phi_4 \\
\boxed{\sigma_7^\boxtimes G^\boxtimes \sigma_2^\boxtimes} & \xrightarrow[\Phi_2]{\text{via}} & \Delta^\boxtimes H^\boxtimes \sigma_7^\boxtimes G^\boxtimes \sigma_2^\boxtimes & \xrightarrow[\beta_7]{\text{via}} & \Delta^\boxtimes \sigma_6^\boxtimes F^\boxtimes G^\boxtimes \sigma_2^\boxtimes & \xrightarrow[\Phi_6]{\text{via}} & F^\boxtimes G^\boxtimes \sigma_2^\boxtimes
\end{array}$$

Observe that in the top row, upon the canceling of σ_2^\boxtimes on the left and G^\boxtimes on the right from each vertex, the resulting sequence composes to Ψ_{F, σ_4} . Therefore, by Lemma 4.2.6, the composition $\sigma_2^\boxtimes \sigma_4^\boxtimes G^\boxtimes \xrightarrow{\text{top row}} \sigma_2^\boxtimes F^\boxtimes G^\boxtimes \xrightarrow{\text{via } \Phi_3} \sigma_2^\boxtimes$ is the same as $\sigma_2^\boxtimes(\Phi_{\sigma_4 \star G}) = \sigma_2^\boxtimes(\Phi_{\sigma_1})$. By a similar argument we also see that the composition $\sigma_7^\boxtimes G^\boxtimes \sigma_2^\boxtimes \xrightarrow{\text{bottom row}} F^\boxtimes G^\boxtimes \sigma_2^\boxtimes \xrightarrow{\text{via } \Phi_4} \sigma_2^\boxtimes$ is the same as $\Phi_{\sigma_7 \star G'}(\sigma_2^\boxtimes) = \Phi_{\sigma_3}(\sigma_2^\boxtimes)$.

Since the preceding diagram commutes, from its outer border we deduce the lemma. \square

LEMMA 5.2.3. *Let $F: X \rightarrow Y$ be a P-labeled map and $\sigma: Y \rightrightarrows Y$ a sequence such that $|\sigma| = 1_Y$. Then the following two isomorphisms are equal*

$$F^\boxtimes \sigma^\boxtimes \xrightarrow{F^\boxtimes(\Phi_\sigma)} F^\boxtimes, \quad F^\boxtimes \sigma^\boxtimes \xrightarrow{\Psi_{F, F \star \sigma}} F^\boxtimes.$$

PROOF. We construct a cartesian diagram as follows where Δ is the diagonal map with label P.

$$\begin{array}{ccccccc}
X & \xrightarrow{\Delta} & X^2 & \xrightarrow{Q} & X & \xrightarrow{\sigma'} & X \\
& & \downarrow P & \downarrow p & \downarrow F & \downarrow s & \downarrow F \\
& & X & \xrightarrow{F} & Y & \xrightarrow{\sigma} & Y
\end{array}$$

It suffices to prove that the outer border of the following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
F^\boxtimes \sigma^\boxtimes & \xleftarrow{\text{via } \Phi_{\Delta \star P}} & \Delta^\boxtimes P^\boxtimes F^\boxtimes \sigma^\boxtimes \\
& \searrow \beta_s & \downarrow \text{via } \beta_p^{-1} \\
& & \Delta^\boxtimes Q^\boxtimes F^\boxtimes \sigma^\boxtimes \\
& \swarrow \text{via } \Phi_{\sigma'} & \downarrow \text{via } \beta_s^{-1} \\
& & \Delta^\boxtimes Q^\boxtimes \sigma' F^\boxtimes \\
& \swarrow \text{via } \Phi_{\sigma'} & \downarrow \text{via } \Phi_{\Delta \star Q \star \sigma'} \\
F^\boxtimes & \xleftarrow{\text{via } \Phi_{\Delta \star Q \star \sigma'}} & F^\boxtimes
\end{array}$$

The triangle on the left commutes by 5.2.2. The uppermost triangle commutes by pseudofunctoriality since all the maps involved are P-labeled. Commutativity of \ddagger is obvious. Finally, the lowermost triangle commutes by 5.2.1. \square

Now we are in a position to show that $\Psi_{-, -}$ recovers the base change isomorphisms.

PROPOSITION 5.2.4. (Recovering $\beta_{-, -}$). *Consider a cartesian diagram as follows.*

$$\begin{array}{ccc} X & \xrightarrow{\sigma_4} & Y_1 \\ \sigma_3 \downarrow & & \downarrow \sigma_1 \\ Y_2 & \xrightarrow{\sigma_2} & Z \end{array}$$

Then $\Psi_{\sigma_4 \star \sigma_1, \sigma_3 \star \sigma_2} = \beta_{\sigma_2, \sigma_1}$.

PROOF. Our goal may be rephrased as saying that corresponding to the cartesian diagram shown on the left below where $\rho_1 = \sigma_4 \star \sigma_1$, $\rho_2 = \sigma_3 \star \sigma_2$ and Δ is the diagonal map with label P, the diagram of isomorphisms on the right commutes.

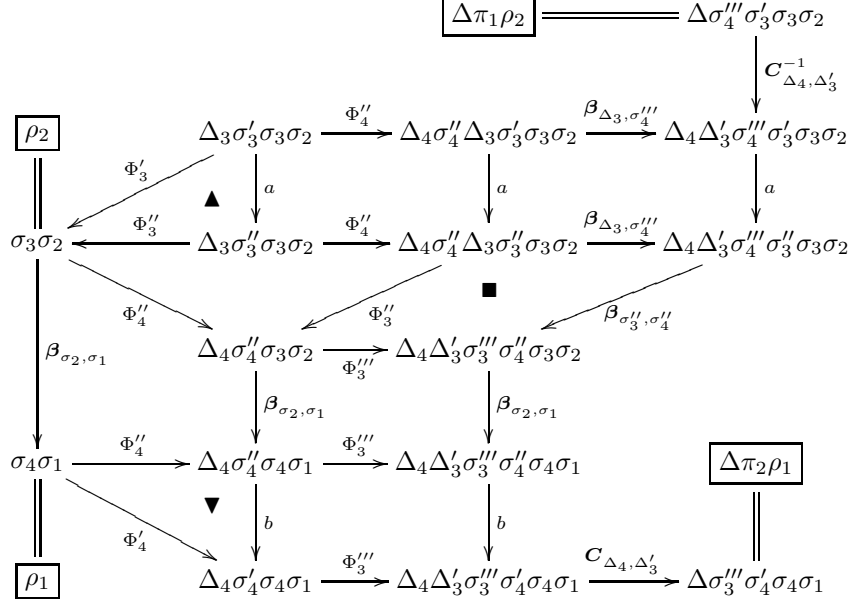
$$(5.2.4.1) \quad \begin{array}{ccc} X & \xrightarrow{\Delta} & X^2 \xrightarrow{\pi_2} X \\ \pi_1 \downarrow & & \downarrow \rho_1 \\ X & \xrightarrow{\rho_2} & Z \end{array} \quad \begin{array}{ccc} \rho_2^{\boxtimes} & \xleftarrow[\Phi_{\Delta \star \pi_1}]{\text{via}} & \Delta^{\boxtimes} \pi_1^{\boxtimes} \rho_2^{\boxtimes} \\ \downarrow \beta_{\sigma_2, \sigma_1} & & \downarrow \beta_{\rho_2, \rho_1} \\ \rho_1^{\boxtimes} & \xleftarrow[\Phi_{\Delta \star \pi_2}]{\text{via}} & \Delta^{\boxtimes} \pi_2^{\boxtimes} \rho_1^{\boxtimes} \end{array}$$

We expand and extend the diagram on the left in (5.2.4.1) as follows. Here the Δ_i 's are the diagonal maps with label P and the rest are determined by the cartesian condition. Note that $|\Delta| = |\Delta_4 \star \Delta'_3|$, $\pi_1 = \sigma_4''' \star \sigma_3'$ and $\pi_2 = \sigma_3''' \star \sigma_4'$.

$$\begin{array}{ccccccc} & & X & & X & & \\ & & \downarrow \Delta_4 & & \downarrow \Delta_4 & & \\ X & \xrightarrow{\Delta_4} & Y_4 & \xrightarrow{\Delta'_3} & X^2 & \xrightarrow{\sigma_3'''} & Y_4 \xrightarrow{\sigma'_4} X \\ \sigma_4'' \downarrow & & \downarrow \sigma_4''' & & \downarrow \sigma_4'' & & \downarrow \sigma_4 \\ X & \xrightarrow{\Delta_3} & Y_3 & \xrightarrow{\sigma_3''} & X & \xrightarrow{\sigma_4} & Y_1 \\ & & \downarrow \sigma_3' & & \downarrow \sigma_3 & & \downarrow \sigma_1 \\ & & X & \xrightarrow{\sigma_3} & Y_2 & \xrightarrow{\sigma_2} & Z \end{array}$$

Correspondingly we expand the diagram on the right in (5.2.4.1) as follows, where the superscript \boxtimes has been omitted for convenience.

(5.2.4.2)



The maps are all isomorphisms. Reduced notation has been employed for all the maps except those denoted by a or b , while the Φ 's are given by the following table.

$$\Phi'_3 = \Phi_{\Delta_3 \star \sigma'_3}, \quad \Phi''_3 = \Phi_{\Delta_3 \star \sigma''_3}, \quad \Phi'''_3 = \Phi_{\Delta'_3 \star \sigma'''_3}^{-1}, \quad \Phi'_4 = \Phi_{\Delta_4 \star \sigma'_4}^{-1}, \quad \Phi''_4 = \Phi_{\Delta_4 \star \sigma''_4}^{-1}.$$

The ones denoted by a are induced in the obvious way by the base-change isomorphism $\sigma'_3 \boxtimes \sigma_3 \xrightarrow{\sim} \sigma''_3 \boxtimes \sigma_3$, while the ones denoted by b are induced in the obvious way by the base-change isomorphism $\sigma'_4 \boxtimes \sigma_4 \xrightarrow{\sim} \sigma''_4 \boxtimes \sigma_4$.

Let us verify that (5.2.4.2) commutes. Of all its subdiagrams, only \blacktriangle , \blacktriangledown and \blacksquare need be considered; the remaining ones commute for functorial reasons.

In \blacksquare , we may cancel off $\Delta_4 \boxtimes$ on the left and $\sigma'_3 \sigma'_2$ on the right, from each vertex. The resulting diagram now commutes by 5.2.2. In \blacktriangle we first cancel σ'_2 on the right. The resulting diagram commutes by 4.6.3(i). By a similar argument \blacktriangledown commutes. Thus (5.2.4.2) commutes.

Therefore, to prove that the diagram on the right in (5.2.4.1) commutes, it suffices to match it with the outer border of (5.2.4.2), comprising of the framed vertices and the maps between them. Here is the list of the requirements wherein three edges from (5.2.4.1) are to be compared with maps obtained from the outer border of (5.2.4.2), the fourth one being shared by both.

(5.2.4.3)

- (i). $\Phi_{\Delta \star \pi_2}(\rho_1^{\boxtimes}) : \Delta^{\boxtimes} \pi_2^{\boxtimes} \rho_1^{\boxtimes} \longrightarrow \rho_1^{\boxtimes} = \text{via outer border of (5.2.4.2) ?}$
- (ii). $\Phi_{\Delta \star \pi_1}(\rho_2^{\boxtimes}) : \Delta^{\boxtimes} \pi_1^{\boxtimes} \rho_2^{\boxtimes} \longrightarrow \rho_2^{\boxtimes} = \text{via outer border of (5.2.4.2) ?}$
- (iii). $\Delta^{\boxtimes}(\beta_{\rho_2, \rho_1}) : \Delta^{\boxtimes} \pi_1^{\boxtimes} \rho_2^{\boxtimes} \longrightarrow \Delta^{\boxtimes} \pi_2^{\boxtimes} \rho_1^{\boxtimes} = \text{via outer border of (5.2.4.2) ?}$

Out of the three, we give proofs for (i) and (ii) below; (iii) follows essentially by functorial considerations.

In (i), ρ_1^\boxtimes is common to each object and moreover, in the corresponding portion of (5.2.4.2), every vertex contains $\rho_1^\boxtimes = \sigma_4^\boxtimes \sigma_1^\boxtimes$. Upon canceling these terms, (i) reduces to checking that the outer border of the following diagram commutes.

$$\begin{array}{ccc}
 \boxed{\Delta^\boxtimes \pi_2^\boxtimes} & \xrightarrow{\Phi_{\Delta * \pi_2}} & \boxed{1_{\mathcal{D}_X}} \\
 \parallel & \uparrow \dagger_1 & \uparrow \dagger_2 \\
 \Delta^\boxtimes \sigma_3''' \sigma_4^\boxtimes & \xrightarrow[\text{via } C_{\Delta_4, \Delta_3}^{-1}]{} & \Delta_4^\boxtimes \Delta_3' \sigma_3''' \sigma_4^\boxtimes \xrightarrow[\text{via } \Phi_{\Delta_3' * \sigma_3'''}]{} \Delta_4^\boxtimes \sigma_4^\boxtimes \\
 & & \uparrow \Phi_{\Delta_4 * \sigma_4'}
 \end{array}$$

Here $\Phi_{\dots} = \Phi_{\Delta_4 * \Delta_3' * \sigma_3''' * \sigma_4'}$. Now, in the bottom row, both the maps can be rewritten as

$$\text{via } C_{\Delta_4, \Delta_3}^{-1} = \text{via } \Psi_{\Delta_4 * \Delta_3', \Delta}, \quad \text{via } \Phi_{\Delta_3' * \sigma_3'''} = \text{via } \Psi_{\Delta_4, \Delta_4 * \Delta_3' * \sigma_3'''},$$

where for the former we use 5.1.2, while for the latter we use 5.2.3. With this description, both \dagger_1 and \dagger_2 are seen to commute by 4.2.6.

In (ii), we begin as in (i) by first canceling the common factor, which in this case is ρ_2^\boxtimes or $\sigma_3^\boxtimes \sigma_2^\boxtimes$. Thereupon, (ii) reduces to checking that the outer border of the following diagram with omitted superscripts \boxtimes commutes.

$$\begin{array}{ccccc}
 \boxed{\Delta \pi_1} & \xrightarrow{\Phi} & & \boxed{1_{\mathcal{D}_X}} & \\
 \searrow & \uparrow \dagger_1 & & \uparrow \Phi & \swarrow \Phi \\
 \Delta \sigma_4''' \sigma_3' & \xrightarrow[\text{via } C_{\Delta_4, \Delta_3}^{-1}]{} & \Delta_4 \Delta_3' \pi_1 & \xrightarrow{\Phi} & \Delta_3 \sigma_3' \\
 \searrow & \uparrow \dagger_2 & \uparrow \dagger_3 & \uparrow \dagger_4 & \swarrow \\
 & \Delta_4 \Delta_3' \sigma_4''' \sigma_3' & \xrightarrow[\text{via } \beta_{\sigma_4''', \Delta_3}]{} & \Delta_4 \sigma_4'' \Delta_3 \sigma_3' & \\
 & & & \uparrow \Phi & \\
 & & & \Delta_4 \sigma_4'' \Delta_3 \sigma_3' &
 \end{array}$$

In the four places where Φ occurs without a subscript, the missing subscript is the entire sequence given by the source of the corresponding map. By 4.2.6, \dagger_1 commutes while \dagger_2 commutes for functorial reasons. For \dagger_3 , we use 4.2.7 while \dagger_4 commutes by 5.2.1. Thus (5.2.4.3)(ii) is verified.

This concludes the proof of the commutativity of the diagram on the right in (5.2.4.1). \square

5.3. Linearity. We show that $\Psi_{-, -}$ is linear in the following sense.

THEOREM 5.3.1. *Let $\sigma_i: X \Rightarrow Y$ for $i = 1, 2$ be two sequences such that $|\sigma_1| = |\sigma_2|$. Then for any two sequences $\rho: W \Rightarrow X$ and $\tau: Y \Rightarrow Z$, the following two isomorphisms are equal*

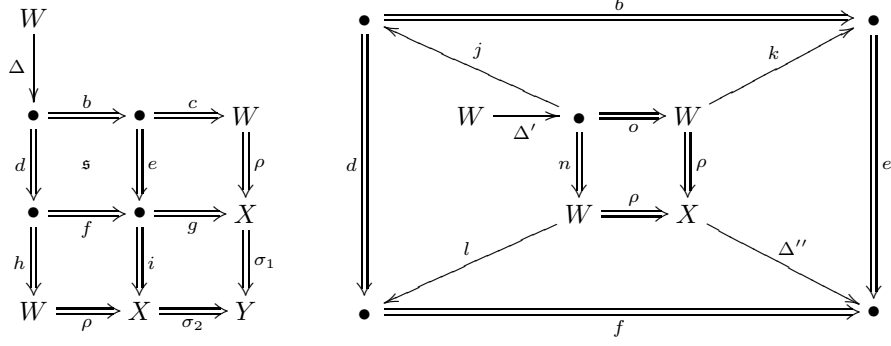
$$\rho^\boxtimes \sigma_2^\boxtimes \tau^\boxtimes \xrightarrow{\Psi_{\rho * \sigma_1 * \tau, \rho * \sigma_2 * \tau}} \rho^\boxtimes \sigma_1^\boxtimes \tau^\boxtimes, \quad \rho^\boxtimes \sigma_2^\boxtimes \tau^\boxtimes \xrightarrow{\text{via } \Psi_{\sigma_1, \sigma_2}} \rho^\boxtimes \sigma_1^\boxtimes \tau^\boxtimes.$$

PROOF. It suffices to prove the theorem for the two special cases when either ρ or τ is empty. In other words, it suffices to prove that the following hold

$$\Psi_{\rho * \sigma_1, \rho * \sigma_2} = \rho^\boxtimes(\Psi_{\sigma_1, \sigma_2}), \quad \Psi_{\sigma_1 * \tau, \sigma_2 * \tau} = \Psi_{\sigma_1, \sigma_2}(\tau^\boxtimes).$$

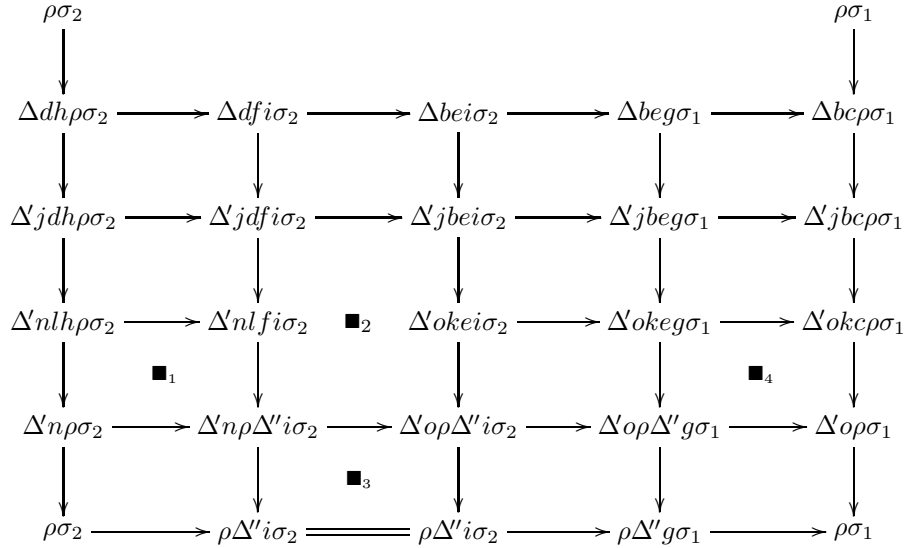
We only show the first of these (linearity on the left). The second one is proved using similar methods.

Consider the following two cartesian diagrams.



The diagram on the right is a cube built on \mathfrak{s} , the northwestern square in the diagram on the left. Thus, the cube is obtained as a fibered product of Δ'' with \mathfrak{s} . (As usual, $\Delta, \Delta', \Delta''$ are diagonal maps with label P.) In particular, $|\Delta' \star j| = |\Delta|$.

Consider the following diagram of isomorphisms where the common superscript \boxtimes has been omitted for convenience.



The morphisms are all obvious ones induced by C , β or Φ for suitable superscript that can be determined in an obvious way for each edge. Using this diagram one deduces the theorem as follows.

First we verify that the preceding diagram commutes. The unnamed rectangles commute for functorial reasons. In \blacksquare_1 , upon canceling $\Delta'' n^{\boxtimes}$ on the left and σ_2^{\boxtimes} on the right from each vertex, we are left with a diagram whose commutativity follows from 5.2.2 using the following substitutions.

$$\sigma_1 \rightsquigarrow \Delta'' \star i, \quad \sigma_3 \rightsquigarrow l \star h, \quad \sigma_2 \rightsquigarrow \rho, \quad \text{etc.}$$

By a similar argument \blacksquare_4 commutes. For \blacksquare_2 we use the cube lemma. Finally, for \blacksquare_3 , we first cancel $\Delta'' i^{\boxtimes} \sigma_2^{\boxtimes}$ on the right and then conclude by the reflexivity property $\Psi_{\rho, \rho} = \text{identity}$.

Now note that the topmost row of maps in the preceding diagram defines $\Psi_{\rho \star \sigma_1, \rho \star \sigma_2}$ while the bottommost row defines $\rho^\boxtimes(\Psi_{\sigma_1, \sigma_2})$. Therefore to prove the theorem, it suffices to verify the leftmost column of maps and the rightmost one, each composes to the corresponding identity transformation. Each of these verifications is carried out in the same way. For instance, for the left column, we may first cancel $\rho^\boxtimes \sigma_2^\boxtimes$ from each object and then use the outer border of the following diagram where each unnamed arrow is Φ_- with suitable subscript.

$$\begin{array}{ccccc}
 \Delta^\boxtimes d^\boxtimes h^\boxtimes & \xrightarrow{\quad} & 1_{\mathcal{D}_X} & \xleftarrow{\quad} & \Delta'^\boxtimes n^\boxtimes \\
 \text{via } C_{\Delta', j} \uparrow & & \nearrow & & \uparrow \text{via } \Phi_{l \star h} \\
 \Delta'^\boxtimes j^\boxtimes d^\boxtimes h^\boxtimes & \xrightarrow{\quad} & & \xleftarrow{\quad} & \Delta'^\boxtimes n^\boxtimes l^\boxtimes h^\boxtimes
 \end{array}$$

The triangle on the left commutes by 4.2.6 (because $C_{\Delta', j} = \Psi_{\Delta, \Delta' \star j}$), the one in the middle commutes by 4.2.7 and the one on the right commutes by 5.2.1. \square

6. Proofs IV (the output)

We give proofs for Theorem 2.2.4, Theorem 2.3.2, and Theorem 2.4.3. The proofs are based on the results of §3–§5.

6.1. Preliminaries on pseudofunctorial covers. We discuss some basic results concerning pseudofunctorial covers (2.2.1) that we shall need in the proof of Theorem 2.2.4. Henceforth, we refer to a pseudofunctorial cover simply as a cover.

6.1.1. We use the following notation. Let $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ be a cover. For any object X in \mathcal{C} , let $S_X : X^! \rightarrow \mathcal{D}_X$ be the associated functor.

- (i) For any labeled map $F = (f, \lambda) : X \rightarrow Y$, we set $F^! := f^!$ and for any labeled sequence σ as follows

$$X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} X_3 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{F_n} X_{n+1},$$

we set $\sigma^! = F_1^! \cdots F_n^!$.

- (ii) To every labeled map F as in (i), we associate a map $S_X F^! \rightarrow F^\boxtimes S_Y$, which, for $\lambda = \mathbf{P}$, is the canonical map $S_X f^! \rightarrow f^\times S_Y$ obtained from $\mu_\times^!$ and for $\lambda = \mathbf{O}$, is the canonical map $S_X f^! \rightarrow f^\square S_Y$ obtained from $\mu_\square^!$. To every sequence σ as in (i), we associate the map $S_{X_1} \sigma^! \rightarrow \sigma^\boxtimes S_{X_{n+1}}$ which is the obvious one obtained by successively using $S_{X_i} F_i^! \rightarrow F_i^\boxtimes S_{X_{i+1}}$.

It follows that if \mathcal{C} is a perfect cover, (2.2.3) then $S_{X_1} \sigma^! \rightarrow \sigma^\boxtimes S_{X_{n+1}}$, as defined in (ii) above, is an isomorphism.

LEMMA 6.1.2. *Let $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ be a cover.*

- (i) *For any labeled sequence σ such that $|\sigma|$ is an identity map, say 1_X , the following diagram of obvious natural maps commutes.*

$$\begin{array}{ccc}
 S_X \sigma^! & \xrightarrow{\quad} & \sigma^\boxtimes S_X \\
 \searrow \text{via } (-)^! & & \swarrow \text{via } \Phi_\sigma \\
 & S_X &
 \end{array}$$

(ii) For any cartesian square \mathcal{S} as follows,

$$\begin{array}{ccc} W & \xrightarrow{\sigma_4} & X \\ \sigma_3 \downarrow & & \downarrow \sigma_1 \\ Z & \xrightarrow{\sigma_2} & Y \end{array}$$

the following diagram of obvious natural maps commutes.

$$\begin{array}{ccc} S_W \sigma_4^! \sigma_1^! & \xrightarrow{\text{via } (-)^!} & S_W \sigma_3^! \sigma_2^! \\ \downarrow & & \downarrow \\ \sigma_4^{\boxtimes} S_X \sigma_1^! & & \sigma_3^{\boxtimes} S_Z \sigma_2^! \\ \downarrow & & \downarrow \\ \sigma_4^{\boxtimes} \sigma_1^{\boxtimes} S_Y & \xrightarrow{\text{via } \beta_s} & \sigma_3^{\boxtimes} \sigma_2^{\boxtimes} S_Y \end{array}$$

PROOF. (Sketch) For simplicity we shall assume that for all X , $X^! = \mathcal{D}_X$ and that S_X is the identity functor $\mathbf{1}_{\mathcal{D}_X}$. The proof in the general case is obtained by inserting S_- 's in the appropriate places.

For (i), one looks at a staircase \mathcal{S} based on σ . Consider the following diagram whose bottom row spells out the definition of Φ_σ as in (3.4.1.5), the remaining maps being the obvious ones.

$$\begin{array}{ccccccc} \sigma^! & \longrightarrow & V_1^! \sigma^! & \longrightarrow & \text{Steps}^! & \longrightarrow & \mathbf{1}_{\mathcal{D}_X} \\ \downarrow & \wr_1 & \downarrow & \wr_2 & \downarrow & \wr_3 & \parallel \\ \sigma^{\boxtimes} & \longrightarrow & V_1^{\boxtimes} \sigma^{\boxtimes} & \longrightarrow & \text{Steps}^{\boxtimes} & \longrightarrow & \mathbf{1}_{\mathcal{D}_X} \end{array}$$

To prove (i) it suffices to check that this diagram commutes.

We may expand \wr_1 as follows.

$$\begin{array}{ccc} \sigma^! & \longrightarrow & V_1^! \sigma^! \\ \downarrow & & \downarrow \\ \sigma^{\boxtimes} & \longrightarrow & V_1^{\boxtimes} \sigma^{\boxtimes} \end{array}$$

(Note: The diagram above is a simplified representation of the trapezium and triangle commutativity shown in the image. The image shows a trapezium with vertices $\sigma^!$, $V_1^! \sigma^!$, $V_1^! \sigma^{\boxtimes}$, and σ^{\boxtimes} , and a triangle with vertices $\sigma^!$, $V_1^! \sigma^!$, and $V_1^{\boxtimes} \sigma^{\boxtimes}$. Arrows indicate the relationships between these objects.)

The trapezium commutes for functorial reasons. Since V_1 consists of \mathbf{P} -labeled maps only, we deduce from the pseudofunctoriality of the isomorphism $\mu_X^!$ that the triangle in the preceding diagram also commutes.

In \wr_2 , the bottom arrow is given by base-change isomorphisms associated to each of the $n(n-1)/2$ squares occurring in the staircase \mathcal{S} . By using the compatibility in 2.2.1(b) for each of these squares one obtains that \wr_2 commutes. In \wr_3 , one uses 2.2.1(c) for each of the n steps in \mathcal{S} to deduce commutativity.

Finally, (ii) is proved using similar arguments. \square

PROPOSITION 6.1.3. *Let $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ be a cover. For any two sequences σ_1, σ_2 such that $|\sigma_1| = |\sigma_2|$, the following diagram of obvious natural maps commutes where X is the source of σ_i and Y the target.*

$$\begin{array}{ccc} S_X \sigma_1^! & \xrightarrow{\text{via } (-)^!} & S_X \sigma_2^! \\ \downarrow & & \downarrow \\ \sigma_1^\boxtimes S_Y & \xrightarrow{\text{via } \Psi_{\sigma_2, \sigma_1}} & \sigma_2^\boxtimes S_Y \end{array}$$

PROOF. Let us assume for simplicity that for all X , $X^! = \mathcal{D}_X$ and that S_X is the identity functor $\mathbf{1}_{\mathcal{D}_X}$. We use the diagram defining $\Psi_{\sigma_1, \sigma_2}$ (see 4.1.1). It suffices to check that the following diagram commutes.

$$\begin{array}{ccccccc} \sigma_2^! & \longrightarrow & \Delta^! \sigma_1^! \sigma_2^! & \longrightarrow & \Delta^! \sigma_2^! \sigma_1^! & \longrightarrow & \sigma_1^! \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \sigma_2^\boxtimes & \longrightarrow & \Delta^\boxtimes \sigma_1^\boxtimes \sigma_2^\boxtimes & \longrightarrow & \Delta^\boxtimes \sigma_2^\boxtimes \sigma_1^\boxtimes & \longrightarrow & \sigma_1^\boxtimes \end{array}$$

The square in the middle commutes by 6.1.2(ii) while the commutativity of the squares on the either ends follows easily from 6.1.2(i). \square

6.2. Proof of Theorem 2.2.4.

6.2.1. Proof of part (i): We construct a perfect cover $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ as follows.

For any object X in \mathcal{C} , set $X^! := \mathcal{D}_X$ and set $S_X := \mathbf{1}_{\mathcal{D}_X}$. For every map f in \mathcal{Q} , choose a labeled sequence σ_f such that $|\sigma_f| = f$ with the understanding that if f is an identity map, say 1_X , then σ_f is a single map, viz., $(1_X, \mathbf{P})$. For any map f in \mathcal{Q} , set $f^! := \sigma_f^\boxtimes$. For any pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{Q} , set $C_{f,g}^! = \Psi_{\sigma_{gf}, \sigma_f \star \sigma_g}$.

Let us verify that $(-)^!$ is a pseudofunctor. We only check the associativity property of $C_{-, -}^!$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be maps in \mathcal{Q} . Our aim is to verify that

$$C_{gf,h}^! \circ C_{f,g}^! (h^!) = C_{f,hg}^! \circ f^! (C_{g,h}^!).$$

By definition, this amounts to verifying that

$$(6.2.1.1) \quad \Psi_{\sigma_{hgf}, \sigma_{gf} \star \sigma_h} \circ \Psi_{\sigma_{gf}, \sigma_f \star \sigma_g} (\sigma_h^\boxtimes) = \Psi_{\sigma_{hgf}, \sigma_f \star \sigma_{hg}} \circ \sigma_f^\boxtimes (\Psi_{\sigma_{hg}, \sigma_g \star \sigma_h}).$$

To that end, we verify that each side equals $\Psi_{\sigma_{hgf}, \sigma_f \star \sigma_g \star \sigma_h}$. For the left-hand side, we see that by 5.3.1,

$$\Psi_{\sigma_{gf}, \sigma_f \star \sigma_g} (\sigma_h^\boxtimes) = \Psi_{\sigma_{gf} \star \sigma_h, \sigma_f \star \sigma_g \star \sigma_h}$$

and therefore, by the cocycle rule, (4.2.1) the left-hand side of (6.2.1.1) is the same as $\Psi_{\sigma_{hgf}, \sigma_f \star \sigma_g \star \sigma_h}$. A similar argument works for the right-hand side. Thus $C_{-, -}^!$ is associative.

Next we define on \mathbf{P} , a pseudofunctorial isomorphism $\mu_\times^! : (-)^!|_{\mathbf{P}} \xrightarrow{\sim} (-)^\times$ as follows. For any $f : X \rightarrow Y$ in \mathbf{P} , we define the required map $f^! \xrightarrow{\sim} f^\times$ to be Ψ_{F, σ_f} where $F := (f, \mathbf{P})$. For this to give a pseudofunctorial isomorphism it remains to be verified that for any pair of \mathbf{P} -morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the following diagram

commutes where $F = (f, P), G = (g, P), H = (gf, P)$.

$$\begin{array}{ccccc}
 f^!g^! & \xrightarrow[\text{via } \Psi_{F,\sigma_f}]{a} & f^\times g^! & \xrightarrow[\text{via } \Psi_{G,\sigma_g}]{b} & f^\times g^\times \\
 \Psi_{\sigma_{gf},\sigma_f \star \sigma_g} \downarrow & & & & c \downarrow C_{f,g}^\times \\
 (gf)^! & \xrightarrow{\Psi_{H,\sigma_{gf}}} & & & (gf)^\times
 \end{array}$$

By 5.3.1, the maps a and b in the top row equal $\Psi_{F \star \sigma_g, \sigma_f \star \sigma_g}$ and $\Psi_{F \star G, F \star \sigma_g}$ respectively. By 5.1.2, c equals $\Psi_{H, F \star G}$. Thus all the maps in the preceding diagram can be expressed as $\Psi_{-, -}$ suitably. Now we check that each of the two paths from $f^!g^!$ to $(gf)^\times$ composes to $\Psi_{H, \sigma_f \star \sigma_g}$ by repeatedly using the cocycle condition. This proves that $\mu_\times^!$ is a pseudofunctorial map.

The remaining verifications needed to complete the construction of \mathcal{C} as a cover, such as defining $\mu_\square^!$ and checking that (b) and (c) of 2.2.1 hold, are carried out in a similar manner. For verifying commutativity of any diagram encountered, one uses 5.3.1 or the Propositions in §5.1 and §5.2 to first rewrite every isomorphism encountered as $\Psi_{-, -}$ with suitable subscripts. Then, repeatedly applying the cocycle rule gives the desired commutativity.

6.2.2. Proof of part (ii): Let $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ and $\mathcal{C}' = ((-)^{\#}, \mu_\times^{\#}, \mu_\square^{\#})$. We first show the existence of a map $\mathcal{C}' \rightarrow \mathcal{C}$.

For any object X in \mathcal{C} , let $S_X^!: X^! \rightarrow \mathcal{D}_X$ and $S_X^{\#}: X^{\#} \rightarrow \mathcal{D}_X$ be the functors associated to \mathcal{C} and \mathcal{C}' respectively. Let $T_X: X^{\#} \rightarrow X^!$ be the unique functor such that $S_X^! T_X = S_X^{\#}$. Such a functor exists because, by hypothesis $S_X^!$ is an isomorphism.

For any labeled sequence $\sigma: X \rightarrow Y$ we define $\tau_\sigma: T_X \sigma^{\#} \rightarrow \sigma^! T_Y$ to be the unique map making the following diagram commute where the remaining maps are the obvious ones resulting from $\mathcal{C}, \mathcal{C}'$ being covers. (see 6.1.1)

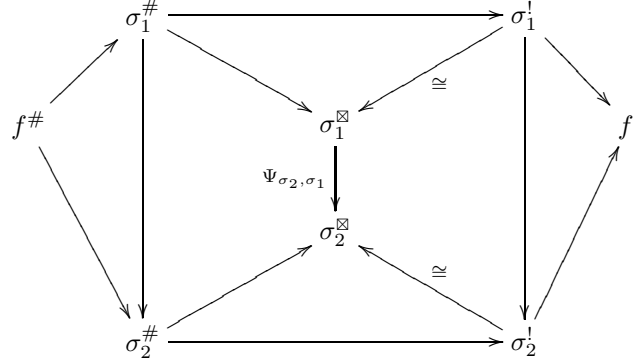
$$\begin{array}{ccccc}
 S_X^! T_X \sigma^{\#} & \xrightarrow{S_X^! (\tau_\sigma)} & S_X^! \sigma^! T_Y & \xrightarrow{\cong} & \sigma^{\boxtimes} S_Y^! T_Y \\
 \parallel & & & & \parallel \\
 S_X^{\#} \sigma^{\#} & \xrightarrow{\quad \quad \quad} & \sigma^{\boxtimes} S_X^{\#} & &
 \end{array}$$

In particular, if σ consists of a single labeled map (f, P) , then the associated map $T_X f^{\#} \rightarrow f^! T_Y$ is the unique one determined by the commutativity of the diagram on the left in 2.2.2; the analogous statement holds for $\sigma = (f, O)$.

Before proceeding further we make a simplification. We shall now assume that for any object X in \mathcal{C} , it holds that $X^{\#} = X^! = \mathcal{D}_X$ and that each of $S_X^!, S_X^{\#}, T_X$ is the identity functor $\mathbf{1}_{\mathcal{D}_X}$. The proof in the general case is only a trivial but notationally cumbersome modification of what we give below.

In the simplified setup, we have so far defined, for every labeled sequence σ , a map $\sigma^{\#} \rightarrow \sigma^!$. Now consider the case of a general map f in \mathcal{Q} . Let σ be any labeled sequence such that $|\sigma| = f$. Then we obtain a map $f^{\#} \xrightarrow{\sim} \sigma^{\#} \rightarrow \sigma^! \xrightarrow{\sim} f^!$ where the isomorphisms at either end are the canonical ones resulting from pseudofunctoriality. Our immediate aim is to verify that this map is independent of the choice of σ .

To that end, let σ_1, σ_2 be two sequences such that $|\sigma_i| = f$. It suffices to verify that the outer border of the following diagram of obvious natural maps commutes.



The two triangles on the left and right sides commute by pseudofunctoriality. The two trapeziums commute by Proposition 6.1.3. The remaining two triangles commute by definition.

It remains to verify that we now have a pseudofunctorial map $(-)^{\#} \rightarrow (-)^{!}$ on \mathbf{Q} . To that end, let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in \mathbf{Q} . It suffices to verify that the following diagram commutes.

$$\begin{array}{ccc} f^{\#}g^{\#} & \longrightarrow & f^{!}g^{!} \\ \downarrow & & \downarrow \\ (gf)^{\#} & \longrightarrow & (gf)^{!} \end{array}$$

Let $\sigma_f, \sigma_g, \sigma_{gf}$ be sequences such that $|\sigma_f| = f$, $|\sigma_g| = g$ and $|\sigma_{gf}| = gf$. Then, using the definition of the horizontal maps in the preceding diagram we expand it as follows.

$$\begin{array}{ccccccccc} f^{\#}g^{\#} & \longrightarrow & \sigma_f^{\#}\sigma_g^{\#} & \longrightarrow & \sigma_f^{\boxtimes}\sigma_g^{\boxtimes} & \longrightarrow & \sigma_f^{!}\sigma_g^{!} & \longrightarrow & f^{!}g^{!} \\ \downarrow & & \downarrow & & \downarrow \Psi_{\sigma_{gf}, \sigma_f \star \sigma_g} & & \downarrow & & \downarrow \\ (gf)^{\#} & \longrightarrow & \sigma_{gf}^{\#} & \longrightarrow & \sigma_{gf}^{\boxtimes} & \longrightarrow & \sigma_{gf}^{!} & \longrightarrow & (gf)^{!} \end{array}$$

The two squares on either end commute for pseudofunctorial reasons. The ones in the middle commute by Proposition 6.1.3 for $\sigma_1 = \sigma_f \star \sigma_g$ and $\sigma_2 = \sigma_{gf}$.

Thus we have shown the existence of a map of covers $\mathcal{C}' \rightarrow \mathcal{C}$. Now let us verify that there is a unique choice for such a map. Once again we assume, for simplicity, that for any X , each of $S_X^!, S_X^{\#}, T_X$ is the identity functor $\mathbf{1}_{\mathcal{D}_X}$. Let $\tau_i: \mathcal{C}' \rightarrow \mathcal{C}$ for $i = 1, 2$ be maps of covers. Now recall that for any map f in \mathbf{O} or \mathbf{P} , the choice for a map $f^{\#} \rightarrow f^{!}$ is uniquely determined by the commutativity of the corresponding diagram in 2.2.2. Therefore, on \mathbf{O} as well as on \mathbf{P} , the restrictions of τ_i coincide. Since any map in \mathbf{Q} is a composite of maps in \mathbf{O} or \mathbf{P} , the pseudofunctoriality of the map $\mathcal{C}' \rightarrow \mathcal{C}$ implies uniqueness in general.

6.3. Proof of flat-base-change theorem.

6.3.1. We may upgrade the flat base-change isomorphisms β_- of §2.3[E3](i),(ii) to operate on the level of labeled maps (with label O or P) and labeled sequences in the obvious manner. For any diagram \mathfrak{s} as follows,

$$\begin{array}{ccc} W & \xrightarrow{u'} & X \\ G' \downarrow & & \downarrow G \\ Z & \xrightarrow{u} & Y \end{array}$$

where G, G' are labeled maps having the same label, $u, u' \in \mathbf{F}$, and the underlying diagram is a fibered square, we may associate a base-change isomorphism $\beta_{\mathfrak{s}}: u'^b F^{\boxtimes} \xrightarrow{\sim} F'^{\boxtimes} u^b$ that is defined in the expected manner. More generally, for a diagram \mathfrak{S} as follows, where σ_1, σ'_1 are labeled sequences, $u, u' \in \mathbf{F}$, and the underlying diagram of ordinary maps is cartesian,

$$(6.3.1.1) \quad \begin{array}{ccc} W & \xrightarrow{u'} & X \\ \sigma' \Downarrow & & \Downarrow \sigma \\ Z & \xrightarrow{u} & Y \end{array}$$

we associate a base-change isomorphism $\beta_{\mathfrak{S}}: u'^b \sigma^{\boxtimes} \xrightarrow{\sim} \sigma'^{\boxtimes} u^b$, (also to be denoted by $\beta_{\sigma, u}$) that is defined via the β_- 's associated to each unit square in it. Note that β_- is transitive vertically and horizontally.

Any diagram, such as the preceding one shall henceforth be also called cartesian.

PROPOSITION 6.3.2. *Consider two cartesian diagrams as follows where $u \in \mathbf{F}$, σ_i are labeled sequences and $|\sigma_1| = |\sigma_2|$, $|\sigma'_1| = |\sigma'_2|$.*

$$\begin{array}{ccc} W & \xrightarrow{u'} & X \\ \sigma'_1 \Downarrow & & \Downarrow \sigma_1 \\ Z & \xrightarrow{u} & Y \end{array} \quad \begin{array}{ccc} W & \xrightarrow{u'} & X \\ \sigma'_2 \Downarrow & & \Downarrow \sigma_2 \\ Z & \xrightarrow{u} & Y \end{array}$$

Then the following diagram commutes.

$$\begin{array}{ccc} u'^b \sigma_1^{\boxtimes} & \xrightarrow{\beta_{\sigma_1, u}} & \sigma_1'^{\boxtimes} u^b \\ \text{via } \Psi_{\sigma_2, \sigma_1} \downarrow & & \downarrow \text{via } \Psi_{\sigma_2', \sigma_1'} \\ u'^b \sigma_2^{\boxtimes} & \xrightarrow{\beta_{\sigma_2, u}} & \sigma_2'^{\boxtimes} u^b \end{array}$$

PROOF. The proof uses the same kind of arguments we have been using so far and so we only give a sketch. We proceed in three steps.

Step 1. For the cartesian square of (6.3.1.1), we claim that under the further assumption that $X = Y$, $W = Z$, $|\sigma| = 1_X$, $|\sigma'| = 1_Z$ and $u = u'$, the following diagram commutes.

$$\begin{array}{ccc} u^b \sigma^{\boxtimes} & \xrightarrow{\beta_{\sigma, u}} & \sigma'^{\boxtimes} u^b \\ & \searrow \text{via } \Phi_{\sigma} & \swarrow \text{via } \Phi_{\sigma'} \\ & u^b & \end{array}$$

The proof of this proceeds along the same lines as 5.2.2.

Step 2. We claim that the cube lemma (4.2.5) also holds when the edges of the cube along a particular direction are all in \mathbf{F} and for the corresponding faces we use β instead of β . Put differently, for the cartesian cube in §2.3[E3](a), if we replace f, f_i, g, g_i by labeled sequences $\sigma, \sigma_i, \rho, \rho_i$ respectively, then the corresponding hexagon also commutes. This claim is proven by induction on the length of these labeled sequences. The basis of induction is the case when all the sequences have length one. If σ and ρ have different labels, then we conclude using [E3](a). If σ and ρ have the same label, then we refer to the first half of the proof of the cube lemma. For the general induction argument we refer to the second half of the proof of the cube lemma.

Step 3. To prove the Proposition, one now looks at the diagram through which $\Psi_{\sigma_2, \sigma_1}$ is defined (see 4.1.1). Upon taking a fibered product of this diagram with u one obtains a 3-dimensional diagram (with a cube in it) relating the the definition of $\Psi_{\sigma_2, \sigma_1}$ with that of $\Psi_{\sigma'_2, \sigma'_1}$. Then one uses Steps 1 and 2 above, upon which the Proposition follows. \square

6.3.3. Proof of Theorem 2.3.2. The proof proceeds along expected lines. First we lift the cartesian diagram \mathfrak{s} of the theorem to a cartesian diagram \mathfrak{S} such as in (6.3.1.1) such that $|\sigma| = f$ and $|\sigma'| = f'$. (see 4.4.2) Now we define $\beta_{\mathfrak{s}}^!$ via

$$u'^b f^! \xrightarrow{\sim} u'^b \sigma^! \xrightarrow{\sim} u'^b \sigma^{\boxtimes} \xrightarrow{\beta_{\mathfrak{s}}} \sigma'^{\boxtimes} u^b \xrightarrow{\sim} \sigma'^! u^b \xrightarrow{\sim} f'^{\boxtimes} u^b.$$

To check that this is independent of the choice of σ it suffices to check that the following diagram obtained via two liftings \mathfrak{S}_1 and \mathfrak{S}_2 of \mathfrak{s} , with corresponding obvious notation, commutes.

$$\begin{array}{ccccccccc} u'^b f^! & \longrightarrow & u'^b \sigma_1^! & \longrightarrow & u'^b \sigma_1^{\boxtimes} & \longrightarrow & \sigma_1'^{\boxtimes} u^b & \longrightarrow & \sigma_1'^! u^b & \longrightarrow & f'^{\boxtimes} u^b \\ & \searrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \nearrow \\ & & u'^b \sigma_2^! & \longrightarrow & u'^b \sigma_2^{\boxtimes} & \longrightarrow & \sigma_2'^{\boxtimes} u^b & \longrightarrow & \sigma_2'^! u^b & & \end{array}$$

The two triangles on either sides commute by pseudofunctoriality, the square in the middle commutes by 6.3.2 and the remaining squares commute for functorial reasons.

The verification that $\beta_-^!$ is transitive along both directions is straightforward to verify. The diagrams in (ii) commute essentially by definition, since for f in \mathbf{P} or \mathbf{O} we may choose σ to be (f, \mathbf{P}) or (f, \mathbf{O}) accordingly.

To check that $\beta_-^!$ is uniquely determined, first note that if f is in \mathbf{P} or \mathbf{O} , then the choice for $\beta_{\mathfrak{s}}^!$ is uniquely determined via (ii). In the general case, since f is a composite of maps in \mathbf{P} and \mathbf{O} , vertical transitivity of $\beta_-^!$ as required in (i) uniquely determines $\beta_{\mathfrak{s}}^!$.

For the last statement of the theorem, one first verifies the commutativity in question for the case when f is in \mathbf{O} or \mathbf{P} . In the general case, one then factors f as a sequence of maps in \mathbf{O} and \mathbf{P} .

6.4. Proof of the Variant Theorem.

We give a proof of Theorem 2.4.3. We first show that the input conditions [A]–[D] and [E1]–[E3] are satisfied, and then use Theorems 2.2.4 and 2.3.2.

In view of 2.4.1(1) and 2.4.1(2), only [D] and [E3](a),(b) need to be verified. For [E3](a) we refer to the first half of the proof of the cube lemma.

Let $X \xrightarrow{f} Y \xrightarrow{g} X$ be maps in \mathbf{C} where $f \in \mathbf{P}$, $g \in \mathbf{O}$ and $gf = 1_X$. Then g is an isomorphism by 2.4.1(4), and hence by 2.4.1(3), we have $g^\square = g^\times$. We define the fundamental isomorphism $\phi_{g,f}: f^\times g^\square \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}_X}$ to be the composition of the following canonical isomorphisms

$$f^\times g^\square = f^\times g^\times \xrightarrow{C_{f,g}^\times} (1_X)^\times = \mathbf{1}_{\mathcal{D}_X}.$$

To verify [D](i)(a), we expand the diagram in question as follows.

$$\begin{array}{ccccc} f'^\times g'^\square h^\times & \longrightarrow & f'^\times h'^\times g^\square & \longrightarrow & h^\times f^\times g^\square \\ \parallel & & \parallel & & \parallel \\ f'^\times g'^\times h^\times & \longrightarrow & f'^\times h'^\times g^\times & \longrightarrow & h^\times f^\times g^\times \\ \downarrow & & & & \downarrow \\ \mathbf{1}_{\mathcal{D}_X}, h^\times & \xlongequal{\quad} & h^\times & \xlongequal{\quad} & h^\times \mathbf{1}_{\mathcal{D}_X} \end{array}$$

The bottom portion commutes by pseudofunctoriality. The square on the upper right corner commutes for functorial reasons. In the square on the upper left corner we may first cancel f'^\times on the left from each vertex and then conclude using 2.4.1(5).

In case of [D](i)(b) we expand the diagram in question as follows.

$$\begin{array}{ccccc} f'^\times g'^\square h^\square & \longrightarrow & f'^\times h'^\square g^\square & \longrightarrow & h^\square f^\times g^\square \\ \parallel & & \parallel & & \parallel \\ f'^\times g'^\times h^\square & \longrightarrow & f'^\times h'^\square g^\times & \longrightarrow & h^\square f^\times g^\times \\ \downarrow & & & & \downarrow \\ \mathbf{1}_{\mathcal{D}_X}, h^\square & \xlongequal{\quad} & h^\square & \xlongequal{\quad} & h^\square \mathbf{1}_{\mathcal{D}_X} \end{array}$$

The top left square commutes by 2.4.1(5), while the top right one commutes for functorial reasons. The bottom portion commutes by transitivity of base-change.

The remaining compatibilities, [D](ii) and [E3](b) are also easy to prove and we leave them for the reader to verify.

Now we verify that in this setup, the notion of a cover as in 2.2.1 is the same as that in 2.4.2. Let $\mathcal{C} = ((-)^!, \mu_\times^!, \mu_\square^!)$ be a cover in the sense of 2.2.1 and let $g: Y \rightarrow X$ be an isomorphism in \mathbf{C} . Let $f: X \rightarrow Y$ be the inverse isomorphism. In view of our construction of the fundamental isomorphism $\phi_{f,g}$, by 2.2.1(c), the outer border of the following diagram commutes.

$$\begin{array}{ccc} S_X f^! g^! & \xrightarrow{\quad} & S_X \mathbf{1}_{X^!} \\ \downarrow & & \parallel \\ f^\times S_Y g^! & \searrow & \\ \downarrow & & \\ f^\times g^\square S_X & \xlongequal{\quad} & f^\times g^\times S_X \longrightarrow \mathbf{1}_{\mathcal{D}_X} S_X \end{array}$$

The pentagon commutes because $(-)^! \rightarrow (-)^\times$ is a map of pseudofunctors. Thus the triangle commutes. Since f^\times is an isomorphism, hence the maps $S_Y g^! \rightarrow g^\times S_X$

and $S_Y g^! \rightarrow g^\square S_X$ are equal. Thus \mathcal{C} is also a cover in the sense of 2.4.2. Converse follows by reversing the above arguments.

Now Theorem 2.4.3 follows from Theorems 2.2.4 and 2.3.2.

7. Applications

We give some applications of our abstract pasting results. In §7.1, we discuss pseudofunctorial properties of the torsion twisted-inverse-image functor $(-)^!$ of Grothendieck duality over noetherian formal schemes. In §7.2, by working with larger derived categories we correspondingly obtain a *prepseudofunctorial* extension of $(-)^!$. In 7.3, we use Lipman's results on duality for non-noetherian formal schemes to define $(-)^!$ over a suitable non-noetherian setup. In §7.4, we look at Huang's construction in [3] of a pseudofunctor $(-)^{\#}$ of zero-dimensional modules over the category of residually finitely generated maps of noetherian complete local rings. In §7.5, we compare our pasting result with Deligne's result in [1].

7.1. Upper shriek for noetherian formal schemes. Let us recall the setup of Grothendieck duality over noetherian formal schemes. We phrase it in terms of our abstract input data.

7.1.1. Set

- \mathbf{C} = The category of morphisms of noetherian formal schemes;
- \mathbf{O} = The subcategory of open immersions in \mathbf{C} ;
- \mathbf{P} = The subcategory of pseudoproper maps in \mathbf{C} ; ([9, 1.2.2])
- \mathbf{F} = The subcategory of flat maps in \mathbf{C} .

For any object X in \mathbf{C} , set $\mathcal{D}_X := \mathbf{D}_{\text{qct}}^+(X)$, the derived category of bounded-below complexes having quasi-coherent torsion \mathcal{O}_X -modules as homology. ([9, §1.2])

Let $f: X \rightarrow Y$ be a map in \mathbf{C} .

If $f \in \mathbf{P}$, then set $f^\times :=$ the right adjoint to $\mathbf{R}f_*: \mathcal{D}_X \rightarrow \mathcal{D}_Y$ ([9, Theorem 2(a)]);

If $f \in \mathbf{O}$, then set $f^\square := \mathbf{L}f^* = f^*$;

If $f \in \mathbf{F}$, then set $f^b := \mathbf{R}\Gamma_X' f^*$ ([9, §1.2]).

The pseudofunctorial structure of $(-)^{\square}$ is the canonical one. For $(-)^{\times}$, we use the obvious one inherited, via adjointness, from the canonical (covariant) pseudofunctor for the derived direct image. In case of $(-)^b$, the comparison map $C_{f,g}^b$ for a pair of \mathbf{F} -maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by the natural map

$$\mathbf{R}\Gamma_X' f^* \mathbf{R}\Gamma_Y' g^* \rightarrow \mathbf{R}\Gamma_X' f^* g^* \xrightarrow{\sim} \mathbf{R}\Gamma_X' (gf)^*.$$

By [9, Proposition 5.2.8(c)], $C_{f,g}^b$ is an isomorphism. Associativity of $C_{-, -}^b$ is easy to verify.

Consider a cartesian square \mathfrak{s} as follows.

$$(7.1.1.1) \quad \begin{array}{ccc} U & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

If $f, g \in \mathbf{P}$ and $i, j \in \mathbf{F}$, then we choose the flat-base-change isomorphism

$$(7.1.1.2) \quad \beta_{\mathfrak{s}}: \mathbf{R}\Gamma_{Uj}^{\prime *} f^{\times} \xrightarrow{\sim} g^{\times} \mathbf{R}\Gamma_V^{\prime *} i^*$$

to be the natural isomorphism of [9, Definition 7.3]. Furthermore, if $i \in \mathbf{O}$, then we choose β_s to be β_s via the canonical identifications $\mathbf{R}L'_U j^* = j^*$ and $\mathbf{R}L'_V i^* = i^*$. The transitivity properties of β_- and β_- follow from the transitivity properties proved in [9, §7.5].

So far we have shown that conditions (1) and (2) of 2.4.1 are achieved. Now we move onto (3) and (4).

Let i be an isomorphism in \mathbf{C} . Then i is in \mathbf{P} as well as in \mathbf{O} and the functors i^* and $\mathbf{R}i_* = i_*$ are both left-adjoint and right-adjoint to each other. We therefore have $i^\square = i^* = i^\times$.

Let $X \xrightarrow{f} Y \xrightarrow{g} X$ be maps in \mathbf{C} where $f \in \mathbf{P}$, $g \in \mathbf{O}$ and $gf = 1_X$. Since g is a surjective as a map of sets $Y \rightarrow X$ therefore it is an isomorphism. In particular, f is also an isomorphism.

Finally 2.4.1(5) is tackled as follows.

LEMMA 7.1.2. *Let \mathbf{s} denote the cartesian square of (7.1.1.1) with $f, g \in \mathbf{P}$ and $i, j \in \mathbf{F}$. If i, j are isomorphisms, then among the following two diagrams the one on the left commutes while if f, g are isomorphisms, then the one on the right commutes.*

$$\begin{array}{ccc} j^* f^\times & \xrightarrow{\beta_s} & g^\times i^* \\ \parallel & & \parallel \\ j^\times f^\times & \xrightarrow{\text{via } (-)^\times} & g^\times i^\times \end{array} \quad \begin{array}{ccc} j^b f^\times & \xrightarrow{\beta_s} & g^\times i^b \\ \parallel & & \parallel \\ j^b f^b & \xrightarrow{\text{via } (-)^b} & g^b i^b \end{array}$$

PROOF. (i). We start with the diagram on the left. Set $\pi := fj = ig$. Then π is in \mathbf{P} . Consider the following diagram where the morphisms are induced via obvious pseudofunctorial ones or via the trace/cotrace maps of the form $\mathbf{R}h_* h^\times \rightarrow \mathbf{1}$ or $\mathbf{1} \rightarrow h^\times \mathbf{R}h_*$ arising from adjointness. (For $h = i$ or $h = j$ these maps are isomorphisms.)

$$\begin{array}{ccccc} & & \boxed{j^* f^\times} & & \\ & \swarrow & \searrow \cong & & \\ g^\times \mathbf{R}g_* j^* f^\times & \xrightarrow{\cong} & g^\times \mathbf{R}g_* \pi^\times & \xleftarrow{\cong} & \pi^\times \xrightarrow{a} \pi^\times \mathbf{R}\pi_* \pi^\times \\ \downarrow \cong & & \downarrow \cong & \searrow \cong & \downarrow b \\ g^\times i^* i_* \mathbf{R}g_* j^* f^\times & \xrightarrow{\cong} & g^\times i^* i_* \mathbf{R}g_* \pi^\times & \xrightarrow{\cong} & g^\times i^* \mathbf{R}\pi_* \pi^\times \\ \downarrow \cong & & \downarrow \cong & \nearrow \cong & \downarrow \\ g^\times i^* \mathbf{R}f_* j_* j^* f^\times & \xrightarrow{\cong} & g^\times i^* \mathbf{R}f_* j_* \pi^\times & \xrightarrow{\cong} & \boxed{g^\times i^*} \xrightarrow{\cong} \pi^\times \\ & \searrow \cong & \nearrow \cong & & \\ & g^\times i^* \mathbf{R}f_* f^\times & & & \end{array}$$

\blacksquare_1 \blacksquare_2

It suffices to check that the preceding diagram commutes because, among the two paths along the outer border between the two framed vertices, the western path gives the base-change isomorphism defined in [9, 7.3], while the eastern one gives the pseudofunctorial isomorphism in view of $ba = 1_{\pi^\times}$ (a formal consequence of adjointness between π^\times and $\mathbf{R}\pi_*$).

Of the subdiagrams, only \blacksquare_i need an explanation since the others commute for functorial reasons. But both $\blacksquare_1, \blacksquare_2$ are exercises in comparing a composition of adjoints with adjoint of a composition; for \blacksquare_1 we first cancel π^\times on the right and for \blacksquare_2 we first cancel $g^\times i^*$ on the left.

(ii). The approach here for the other diagram of the lemma is similar to (i) except for the complications introduced by the presence of the derived torsion functor $\mathbf{R}\Gamma'_-$. Set $\pi := fj = ig$. Then π is in \mathbf{F} . Consider the diagram in (7.1.2.1) below consisting of obvious natural maps where we use $f^\times = f^*$ and $g^\times = g^*$. (Note that some of the functors involved operate only on the larger “non-torsion” categories $\mathbf{D}_{\text{qc}}(-)$ instead of $\mathbf{D}_{\text{qct}}^+(-)$. However the composite functor at each vertex still operates on $\mathbf{D}_{\text{qct}}^+(-)$.)

It suffices to check that (7.1.2.1) commutes since the northern route on the outer border between the two framed vertices gives $\beta_{\mathfrak{s}}$ while the southern route gives the obvious pseudofunctorial isomorphism because $abcd$ is the identity on $\mathbf{R}\Gamma'_U \pi^*$. Commutativity of the subdiagrams are verified easily as in (i) except that for \blacksquare we use commutativity of (7.1.2.2) which consists of isomorphisms and whose bottom row gives the identity on $\mathbf{R}\Gamma'_U$. \square

Having shown that the abstract input conditions 2.4.1(1)–(5) are achieved, we now obtain the output given in Theorem 2.4.3. For convenience, we restate it here in a somewhat self-contained and concise way.

THEOREM 7.1.3. *On the category \mathbf{Q} of composites of compactifiable maps of noetherian formal schemes, there is a contravariant pseudofunctor $(-)^!$, unique up to a unique isomorphism, such that $(-)^!$ takes values in $\mathbf{D}_{\text{qct}}^+(X)$ for any object X in \mathbf{Q} and satisfies the following conditions.*

- (i) *Over the subcategory \mathbf{P} of pseudoproper maps, $(-)^!$ gives the right adjoint to the derived direct-image pseudofunctor $(-)_*$.*
- (ii) *Over the subcategory \mathbf{O} of open immersions, $(-)^!$ is the inverse-image pseudofunctor $(-)^*$.*
- (iii) *For the fibered-product diagram \mathfrak{s} of (7.1.1.1), if i, j are in \mathbf{O} , then the associated base-change isomorphism of (7.1.1.2) equals, via the identifications in (i) and (ii), the obvious pseudofunctorial isomorphism given by $(-)^!$.*

THEOREM 7.1.4. *Let $(-)^!$ be the uniquely determined pseudofunctor of Theorem 7.1.3. Let $(-)^b$ be the natural pseudofunctor on the category \mathbf{F} of flat maps of noetherian formal schemes, that assigns to any \mathbf{F} -map $f: X \rightarrow Y$, the functor $\mathbf{R}\Gamma'_X f^*: \mathbf{D}_{\text{qct}}^+(Y) \rightarrow \mathbf{D}_{\text{qct}}^+(X)$. Then for any fibered-product diagram \mathfrak{s} such as in (7.1.1.1) where f is in \mathbf{Q} and i in \mathbf{F} , there is an associated flat-base-change isomorphism $\beta_{\mathfrak{s}}$ such that the following hold.*

- (i) *If f is in \mathbf{P} , then $\beta_{\mathfrak{s}}$, via the identification in 7.1.3(i), is the isomorphism of (7.1.1.2).*
- (ii) *If f is in \mathbf{O} , then $\beta_{\mathfrak{s}}$, via the identification in 7.1.3(ii), is the pseudofunctorial isomorphism of $(-)^b$ on \mathbf{F} .*
- (iii) *$\beta_{\mathfrak{s}}$ is transitive vis-à-vis horizontal extensions of \mathfrak{s} via \mathbf{F} -maps or vertical extensions via \mathbf{P} -maps.*

REMARK 7.1.5. The pseudofunctor $(-)^!$ of Theorem 7.1.3 also satisfies a universal property that makes it unique up to a unique isomorphism of pseudofunctors.

(7.1.2.1)

$$\begin{array}{ccccccccccc}
 \boxed{j^b f^*} & \xlongequal{\quad} & \mathbf{R}\Gamma'_U j^* f^* & \xleftarrow{\cong} & g^* g_* \mathbf{R}\Gamma'_U j^* f^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V g_* j^* f^* & \xleftarrow{\quad} & g^* \mathbf{R}\Gamma'_V i^* i_* g_* j^* f^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V i^* f_* j_* j^* f^* \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & & \mathbf{R}\Gamma'_U \pi^* & \xleftarrow{\cong} & g^* g_* \mathbf{R}\Gamma'_U \pi^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V g_* \pi^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V i^* i_* g_* \pi^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V i^* f_* j_* \pi^* \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & & \mathbf{R}\Gamma'_U g^* g_* \pi^* & \xleftarrow{\cong} & \mathbf{R}\Gamma'_U g^* i^* i_* g_* \pi^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V i^* \pi_* \pi^* & \xleftarrow{\cong} & g^* \mathbf{R}\Gamma'_V i^* & \xlongequal{\quad} & \boxed{g^* i^b} \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 & & \mathbf{R}\Gamma'_U \pi^* \pi_* \pi^* & \xleftarrow{\cong} & \mathbf{R}\Gamma'_U g^* i^* \pi_* \pi^* & \xleftarrow{\cong} & \mathbf{R}\Gamma'_U g^* i^* & \xleftarrow{\cong} & \mathbf{R}\Gamma'_U \pi^* & \xlongequal{\quad} & \pi^b
 \end{array}$$

$\begin{array}{c} \text{---} \mathbf{R}\Gamma'_U \pi^* \pi_* \pi^* \xleftarrow{a} \mathbf{R}\Gamma'_U g^* i^* \pi_* \pi^* \xleftarrow{b} \mathbf{R}\Gamma'_U g^* i^* \xleftarrow{c} \mathbf{R}\Gamma'_U \pi^* \xleftarrow{d} \pi^b \end{array}$

(7.1.2.2)

$$\begin{array}{ccccc}
 \mathbf{R}\Gamma'_U g^* \mathbf{R}\Gamma'_V g_* & \xrightarrow{\quad} & \boxed{g^* \mathbf{R}\Gamma'_V g_*} & \xrightarrow{\quad} & g^* \mathbf{R}\Gamma'_V g_* \mathbf{R}\Gamma'_U \\
 \downarrow & & \downarrow & & \downarrow \\
 \boxed{\mathbf{R}\Gamma'_U g^* g_*} & \xrightarrow{\quad} & \mathbf{R}\Gamma'_U g^* \mathbf{R}\Gamma'_V g_* \mathbf{R}\Gamma'_U & \xrightarrow{\quad} & \boxed{g^* g_* \mathbf{R}\Gamma'_U} \\
 \downarrow & & \downarrow & & \downarrow \\
 \boxed{\mathbf{R}\Gamma'_U} & \xrightarrow{\quad} & \mathbf{R}\Gamma'_U \mathbf{R}\Gamma'_U & \xrightarrow{\quad} & \boxed{\mathbf{R}\Gamma'_U}
 \end{array}$$

This universal property is precisely of $(-)^!$, as a perfect cover (2.4.2), being final in the category of all covers.

Finally let us show that $(-)^!$ of 7.1.3 extends to étale maps too. Let \mathcal{O}' be the subcategory of \mathcal{C} consisting of formally smooth pseudo-finite-type separated morphisms of relative dimension 0 (see [6, Definition 2.6.2]). For our purposes, we define an étale map to be a map in \mathcal{O}' . For any étale $f: X \rightarrow Y$, set $f^{\square'} = \mathbf{R}\Gamma'_X f^*$. The pseudofunctorial structure of $(-)^{\square'}$ on \mathcal{O}' is the natural one, for instance, the restriction of $(-)^{\flat}$ to \mathcal{O}' . The base-change-isomorphism associated to the fibered product of a P-map with an \mathcal{O}' -map is obtained using β_- of (7.1.1.2).

THEOREM 7.1.6. *The pseudofunctor $(-)^!$ of Theorem 7.1.3 extends to a pseudofunctor on the category \mathcal{Q}' of composites of compactifiable maps and étale maps such that the conditions (i)–(iii) of 7.1.3 hold with \mathcal{O}' in place of \mathcal{O} . Any two such extensions are isomorphic via a unique isomorphism. Finally, the flat-base-change isomorphisms of Theorem 7.1.4 also extend uniquely such that (i)–(iii) of 7.1.4 hold with \mathcal{O}' in place of \mathcal{O} .*

PROOF. The only hurdle in using Theorem 2.4.3 is that condition (4) of 2.4.1 does not hold in this case. The situation is salvaged as follows.

Let $X \xrightarrow{f} Y \xrightarrow{g} X$ be C-maps factoring 1_X such that $f \in \mathcal{P}$ and $g \in \mathcal{O}'$. We claim that Y can be written as a disconnected union $Y = Y_1 \cup Y_2$ such that f is an isomorphism onto Y_1 . Indeed, since g is separated and $gf = 1_X$, f is necessarily a closed immersion. Let \mathcal{I} be the ideal in \mathcal{O}_Y corresponding to f . By [6, Prop. 2.6.8], $\mathcal{I} = \mathcal{I}^2$. Therefore, using the Nakayama lemma over the local rings of Y , we see that the two open subsets $Y_1 := \{y \in Y \mid \mathcal{I}_y = 0\}$ and $Y_2 := \{y \in Y \mid \mathcal{I}_y = \mathcal{O}_{Y,y}\}$ disconnect Y . Clearly f maps X isomorphically to Y_1 .

It follows that the restriction of g to Y_1 is an isomorphism so that on Y_1 we may canonically identify g^\times with $g^{\square'}$. Now we may use the same arguments as in the proof of Theorem 2.4.3 (see §6.4), restricting to suitable connected components whenever necessary, to deduce that in the case of $\mathcal{O}' = \text{étale maps}$, the original input conditions [A]–[D] and [E1]–[E3] are achieved. The theorem then follows. \square

7.2. Prepseudofunctorial extension of upper shriek. We now extend the pseudofunctor $(-)^!$ of Theorem 7.1.3 above by enlarging the category \mathcal{D}_X for each X . What results is no longer a pseudofunctor but a prepseudofunctor in the sense of Lipman, see [4, §1.6]. Let us recall this notion.

7.2.1. A *prepseudofunctor* $(-)^{\#}$ on a category \mathcal{C} has the same data together with conditions as a pseudofunctor except for the following modification: For an object Z in \mathcal{C} we no longer assume that $1_Z^{\#} = \mathbf{1}_{Z^{\#}}$ and instead assign a map $\delta_Z^{\#}: 1_Z^{\#} \rightarrow \mathbf{1}_{Z^{\#}}$, such that for any C-map $X \xrightarrow{f} Y$, the following composites are identity

$$f^{\#} \xrightarrow{(C_{1_X, f}^{\#})^{-1}} 1_X^{\#} f^{\#} \xrightarrow{\text{via } \delta_X^{\#}} f^{\#}, \quad f^{\#} \xrightarrow{(C_{f, 1_Y}^{\#})^{-1}} f^{\#} 1_Y^{\#} \xrightarrow{\text{via } \delta_Y^{\#}} f^{\#}.$$

In particular, $1_Z^{\#}$ is an idempotent functor.

A morphism of prepseudofunctors $(-)^{\#} \rightarrow (-)^!$ on \mathcal{C} is defined the same way as that of pseudofunctors (see beginning of §2) except that we also have the additional

requirement that for any object X in \mathbf{C} , the following diagram commutes.

$$\begin{array}{ccc} S_X 1_X^\# & \xrightarrow{\quad} & 1_X^\dagger S_X \\ & \searrow & \swarrow \\ & S_X & \end{array}$$

7.2.2. We are mainly interested in prepseudofunctors arising out of existing pseudofunctors and idempotent functors in the following way.

Let $(-)^{\#}$ be a pseudofunctor on a category \mathbf{C} . Suppose that for every object X in \mathbf{C} there is a category $X^{\tilde{\#}}$ containing $X^{\#}$ as a full subcategory and that the canonical inclusion $i_X: X^{\#} \rightarrow X^{\tilde{\#}}$ has a right adjoint $\Gamma_X: X^{\#} \rightarrow X^{\tilde{\#}}$ such that the canonical map $\alpha_X: \mathbf{1}_{X^{\#}} \rightarrow \Gamma_X i_X$ induced by adjointness is an isomorphism. For convenience, we suppress all occurrences of the term i_X and also think of Γ_X as a functor $X^{\tilde{\#}} \rightarrow X^{\tilde{\#}}$ that takes $X^{\#}$ to itself. In particular, the canonical map $\beta_X: \Gamma_X \rightarrow \mathbf{1}_{X^{\tilde{\#}}}$ resulting from adjointness, when restricted to $X^{\#}$, gives an isomorphism with inverse α_X . Moreover, for any $A \in X^{\tilde{\#}}$, the composite $\Gamma_X A \xrightarrow{\alpha_X(\Gamma_X A)} \Gamma_X \Gamma_X A \xrightarrow{\Gamma_X(\beta_X(A))} \Gamma_X A$ is the identity. Thus Γ_X is idempotent.

In this setup we obtain a prepseudofunctor $(-)^{\tilde{\#}}$ on \mathbf{C} satisfying the following properties.

- (i) For any map $f: X \rightarrow Y$ in \mathbf{C} , we have $f^{\tilde{\#}} = f^{\#} \Gamma_Y$.
- (ii) For any pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbf{C} , $C_{f,g}^{\tilde{\#}}$ is the composite

$$f^{\tilde{\#}} g^{\tilde{\#}} = f^{\#} \Gamma_Y g^{\#} \Gamma_Z \xrightarrow{\sim} f^{\#} g^{\#} \Gamma_Z \xrightarrow{\sim} (gf)^{\#} \Gamma_Z = (gf)^{\tilde{\#}}.$$

- (iii) For any object X in \mathbf{C} , $\delta_X^{\tilde{\#}}$ is given by $1_X^{\tilde{\#}} = \Gamma_X \xrightarrow{\beta_X} \mathbf{1}_{X^{\tilde{\#}}}$.

Let us verify that $(-)^{\tilde{\#}}$ is indeed a prepseudofunctor. For associativity of $C_{-, -}^{\tilde{\#}}$, we consider maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ and then conclude by looking at the outer border of the following diagram, where the bottommost square on the left commutes by associativity of $C_{-, -}^{\tilde{\#}}$ while the remaining squares commute for functorial reasons.

$$\begin{array}{ccccc} f^{\#} \Gamma_Y g^{\#} \Gamma_Z h^{\#} \Gamma_W & \longrightarrow & f^{\#} g^{\#} \Gamma_Z h^{\#} \Gamma_W & \longrightarrow & (gf)^{\#} \Gamma_Z h^{\#} \Gamma_W \\ \downarrow & & \downarrow & & \downarrow \\ f^{\#} \Gamma_Y g^{\#} h^{\#} \Gamma_W & \longrightarrow & f^{\#} g^{\#} h^{\#} \Gamma_W & \longrightarrow & (gf)^{\#} h^{\#} \Gamma_W \\ \downarrow & & \downarrow & & \downarrow \\ f^{\#} \Gamma_Y (hg)^{\#} \Gamma_W & \longrightarrow & f^{\#} (hg)^{\#} \Gamma_W & \longrightarrow & (hgf)^{\#} \Gamma_W \end{array}$$

For verifying that $\delta^{\tilde{\#}}$ and $C_{-, -}^{\tilde{\#}}$ are compatible we note that for any map $f: X \rightarrow Y$, the following diagrams commute and the bottom rows compose to identity.

$$\begin{array}{ccccccc} f^{\tilde{\#}} & \longrightarrow & 1_X^{\tilde{\#}} f^{\tilde{\#}} & \longrightarrow & f^{\tilde{\#}} & f^{\tilde{\#}} & \longrightarrow & f^{\tilde{\#}} 1_Y^{\tilde{\#}} & \longrightarrow & f^{\tilde{\#}} \\ \parallel & & \parallel & & \parallel & \parallel & & \parallel & & \parallel \\ f^{\#} \Gamma_Y & \longrightarrow & \Gamma_X f^{\#} \Gamma_Y & \longrightarrow & f^{\#} \Gamma_Y & f^{\#} \Gamma_Y & \longrightarrow & f^{\#} \Gamma_Y \Gamma_Y & \longrightarrow & f^{\#} \Gamma_Y \end{array}$$

7.2.3. In the situation of 7.2.2, we say that $(-)^{\tilde{\#}}$ is induced by $(-)^{\#}$ and the family of idempotents Γ_- . We also write $(-)^{\tilde{\#}} := \langle (-)^{\#}, \Gamma_- \rangle$.

Let $(-)^{\tilde{\#}} = \langle (-)^{\#}, \Gamma_-^{\#} \rangle$ and $(-)^{\tilde{!}} = \langle (-)^{!}, \Gamma_-^{!} \rangle$ be prepseudofunctors on \mathbf{C} . Suppose there exists a map $(-)^{\#} \rightarrow (-)^{!}$ and suppose that for any object X in \mathbf{C} , the natural map $S_X: X^{\#} \rightarrow X^{!}$ extends to a map $X^{\tilde{\#}} \rightarrow X^{\tilde{!}}$, also to be denoted as S_X . Then we obtain a Γ -induced map $(-)^{\tilde{\#}} \rightarrow (-)^{\tilde{!}}$, defined as follows.

For any map $f: X \rightarrow Y$ in \mathbf{C} , we define $S_X f^{\tilde{\#}} \rightarrow f^{\tilde{!}} S_Y$ through the following natural maps

$$S_X f^{\tilde{\#}} = S_X f^{\#} \Gamma_Y^{\#} \rightarrow f^{\#} S_Y \Gamma_Y^{\#} \xrightarrow{f^{\#}(\theta_Y)} f^{\#} \Gamma_Y^{!} S_Y = f^{\tilde{!}} S_Y,$$

where θ_Y is the unique map $S_Y \Gamma_Y^{\#} \rightarrow \Gamma_Y^{!} S_Y$ induced via adjointness of $\Gamma_Y^{!}$ by the natural map $S_Y \Gamma_Y^{\#} \rightarrow S_Y$ so that the following diagram of natural maps commutes.

$$\begin{array}{ccc} S_Y \Gamma_Y^{\#} & \xrightarrow{\theta_Y} & \Gamma_Y^{!} S_Y \\ & \searrow & \swarrow \\ & S_Y & \end{array}$$

Since, for any object X , $1_X^{\tilde{\#}} = \Gamma_X^{\#}$ and $1_X^{\tilde{!}} = \Gamma_X^{!}$, the compatibility between $\delta_-^{\tilde{\#}}$ and $\delta_-^{\tilde{!}}$ follows from the defining property of θ_- in the preceding diagram. It only remains to verify that the comparison maps $C_{-, -}^{\tilde{\#}}$ and $C_{-, -}^{\tilde{!}}$ are compatible.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in \mathbf{C} . Then for proving that the following diagram commutes,

$$\begin{array}{ccc} S_X f^{\tilde{\#}} g^{\tilde{\#}} & \xrightarrow{\quad} & S_X (gf)^{\tilde{\#}} \\ \downarrow & & \downarrow \\ f^{\tilde{!}} S_Y g^{\tilde{\#}} & & \\ \downarrow & & \downarrow \\ f^{\tilde{!}} g^{\tilde{!}} S_Z & \xrightarrow{\quad} & (gf)^{\tilde{!}} S_Z \end{array}$$

upon expanding the maps as per their definitions, it suffices to prove that the following diagram of obvious natural maps commutes.

$$\begin{array}{ccccc}
S_X f^\# \Gamma_Y^\# g^\# \Gamma_Z^\# & \longrightarrow & S_X f^\# g^\# \Gamma_Z^\# & \longrightarrow & S_X (gf)^\# \Gamma_Z^\# \\
\downarrow & & \downarrow & & \downarrow \\
f^! S_Y \Gamma_Y^\# g^\# \Gamma_Z^\# & \longrightarrow & f^! S_Y g^\# \Gamma_Z^\# & & \\
\downarrow & \nearrow & \downarrow & & \downarrow \\
f^! \Gamma_Y^! S_Y g^\# \Gamma_Z^\# & & & & (gf)^! S_Z \Gamma_Z^\# \\
\downarrow & & \downarrow & \nearrow & \downarrow \\
f^! \Gamma_Y^! g^! S_Z \Gamma_Z^\# & \longrightarrow & f^! g^! S_Z \Gamma_Z^\# & & \\
\downarrow & & \downarrow & & \downarrow \\
f^! \Gamma_Y^! g^! \Gamma_Z^! S_Z & \longrightarrow & f^! g^! \Gamma_Z^! S_Z & \longrightarrow & (gf)^! \Gamma_Z^! S_Z
\end{array}$$

The triangle commutes by definition of θ_Y and the pentagon commutes because $(-)^{\#} \rightarrow (-)^!$ is a map of pseudofunctors by assumption. The remaining subdiagrams commute for functorial reasons.

7.2.4. The pseudofunctor $(-)^!$ of Theorem 7.1.3 is extended to a prepsseudofunctor as follows. We use the same notation as in 7.1.1. Thus \mathcal{C} consists of noetherian formal schemes, etc..

For any X in \mathcal{C} , set $\tilde{\mathcal{D}}_X := \tilde{\mathbf{D}}_{\text{qc}}^+(X)$, the triangulated subcategory of $\mathbf{D}(X)$ whose objects are complexes \mathcal{F}^\bullet such that $\mathbf{R}\Gamma_X' \mathcal{F}^\bullet \in \mathbf{D}_{\text{qc}}^+(X)$ —or equivalently, $\mathbf{R}\Gamma_X' \mathcal{F}^\bullet \in \mathbf{D}_{\text{qct}}^+(X)$, see [9, 5.2.9]. Set $\Gamma_X^! := \mathbf{R}\Gamma_X'$. Then $\Gamma_X^!$ maps $\tilde{\mathcal{D}}_X$ to \mathcal{D}_X and satisfies the conditions in 7.2.2 above. Thus, the pseudofunctor $(-)^!$ of 7.1.3 induces a prepsseudofunctor $(-)^{\tilde{!}} := \langle (-)^!, \Gamma_-^! \rangle$ on the subcategory $\mathbf{Q} = \overline{\{\mathbf{O}, \mathbf{P}\}}$ of \mathcal{C} .

THEOREM 7.2.5. *Let notation be as in 7.2.4. Let $\mathcal{C} = ((-)^{\natural}, \mu_{\times}^{\natural}, \mu_{\square}^{\natural})$ be a pseudofunctorial cover (in the sense of 2.4.2) and let $(-)^{\tilde{\natural}} = \langle (-)^{\natural}, \Gamma_-^{\natural} \rangle$ be a prepsseudofunctorial extension of $(-)^{\natural}$. Suppose that for ever X in \mathcal{C} , the natural functor $S_X: X^{\natural} \rightarrow \mathcal{D}_X$ given by \mathcal{C} extends to a functor $X^{\tilde{\natural}} \rightarrow \tilde{\mathcal{D}}_X$. Then there exists a unique map of prepsseudofunctors $\epsilon: (-)^{\tilde{\natural}} \rightarrow (-)^{\tilde{!}}$ such that the following diagrams of obvious Γ -induced maps commutes.*

$$\begin{array}{ccc}
(-)^{\tilde{\natural}}|_{\mathbf{P}} & \xrightarrow{\epsilon|_{\mathbf{P}}} & (-)^{\tilde{!}}|_{\mathbf{P}} \\
& \searrow & \swarrow \cong \\
& (-)^{\tilde{\natural}} &
\end{array}
\qquad
\begin{array}{ccc}
(-)^{\tilde{\natural}}|_{\mathbf{O}} & \xrightarrow{\epsilon|_{\mathbf{P}}} & (-)^{\tilde{!}}|_{\mathbf{O}} \\
& \searrow & \swarrow \cong \\
& (-)^{\tilde{\natural}} &
\end{array}$$

PROOF. Over \mathbf{P} and \mathbf{O} , ϵ is determined by the commutativity of the above two diagrams. Since every map in \mathbf{Q} is a composite of maps in \mathbf{O} and \mathbf{P} , uniqueness of ϵ follows. For the existence, the universal property of $(-)^!$ being a perfect cover gives us a map $(-)^{\natural} \rightarrow (-)^!$ and we use its Γ -induced extension for ϵ . \square

REMARK 7.2.6. For applications, it may be desirable to work over a smaller subcategory of \mathcal{C} . Suppose $\overline{\mathcal{C}}$ is a full subcategory of \mathcal{C} such that if Y is an object in $\overline{\mathcal{C}}$, and $f: X \rightarrow Y$ a map in \mathcal{C} , then f is also in $\overline{\mathcal{C}}$ so that X is also an object in $\overline{\mathcal{C}}$. Then $\overline{\mathcal{C}}$ is also stable under fibered products. We may then replace the subcategories $\mathcal{O}, \mathcal{P}, \mathcal{Q}$, etc., by their respective intersections with $\overline{\mathcal{C}}$ to obtain $\overline{\mathcal{O}}, \overline{\mathcal{P}}, \overline{\mathcal{Q}}$, etc. Then we see that the input conditions for pasting still hold over the restricted categories. For the corresponding output, the prepseudofunctorial extensions also carry over. In summary, the conclusion of Theorem 7.2.5 holds over any such restricted setup too.

7.2.7. Flat base change also holds for $(-)^{\bar{!}}$. For a flat map $f: X \rightarrow Y$ we continue to use $f^b := \mathbf{R}\Gamma_X' f^*$ except that f^b is now assumed to act on the larger category $\widetilde{\mathcal{D}}_Y$. Then $(-)^b$ is a prepseudofunctor over \mathbf{F} in the obvious way. For the fibered square in (7.1.1.1), with f in \mathcal{Q} and i in \mathbf{F} , we define the base-change isomorphism $j^b f^{\bar{!}} \xrightarrow{\sim} g^{\bar{!}} i^b$ via

$$j^b f^{\bar{!}} = j^b f^! \Gamma_Y^! \xrightarrow{\sim} g^! i^b \Gamma_Y^! \xleftarrow{\sim} g^! i^b \xleftarrow{\sim} g^! \Gamma_Y^! i^b = g^{\bar{!}} i^b.$$

This flat-base-change isomorphism is also vertically and horizontally transitive. In the case of base change of a proper map, it agrees, via canonical identifications, with the isomorphism in [9, Definition 7.3].

7.3. Upper shriek for non-noetherian ordinary schemes. In [5], Lipman has shown that for many of the constructions of Grothendieck duality over ordinary schemes, the noetherian hypothesis which is frequently used, can be eliminated. What has been lacking is the construction of a pseudofunctor over some category of not-necessarily-noetherian schemes in which both proper maps and étale maps are brought in. We use our abstract pasting results to obtain such a pseudofunctor in the non-noetherian setup. However our results are not entirely satisfactory since we need to impose the additional hypothesis of flatness on the maps in the working category. The problem stems from the fact that the notion of pseudo-coherence of maps, essential for proving many of the duality constructions, is not preserved under base change unless flatness is also assumed.

7.3.1. Here then is the setup as per the input requirements in 2.4.1. We set

$\mathcal{C} = \mathbf{F}$ = The category of flat pseudo-coherent morphisms of quasi-compact quasi-separated schemes; [5, §1]

\mathcal{O} = The subcategory of open immersions in \mathcal{C} ;

\mathcal{P} = The subcategory of proper maps in \mathcal{C} .

Note that every map in \mathcal{C} is quasi-compact [2, I, Cor. 6.1.10]. Also, pseudo-coherence is preserved under flat base change. Hence, both, \mathcal{O} and \mathcal{P} are stable under base change by maps in \mathcal{C} .

For any object X in \mathcal{C} , set $\mathcal{D}_X := \mathbf{D}_{\text{qc}}^+(X)$, the derived category of bounded-below complexes having quasi-coherent \mathcal{O}_X -modules as homology.

Let $f: X \rightarrow Y$ be a map in \mathcal{C} .

If $f \in \mathcal{P}$, then set $f^\times :=$ the right adjoint to $\mathbf{R}f_*: \mathcal{D}_X \rightarrow \mathcal{D}_Y$;

If $f \in \mathcal{O}$, then set $f^\square := \mathbf{L}f^* = f^*$;

If $f \in \mathbf{F}$, then set $f^b := f^*$.

The pseudofunctorial structures for $(-)^{\times}, (-)^{\square}, (-)^{\flat}$ are the obvious ones.

For a cartesian square \mathfrak{s} in \mathbf{C} as follows,

$$(7.3.1.1) \quad \begin{array}{ccc} U & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

if $f, g \in \mathbf{P}$ and $i, j \in \mathbf{F}$, then we choose the flat-base-change isomorphism

$$(7.3.1.2) \quad \mathbb{B}_{\mathfrak{s}}: j^* f^{\times} \xrightarrow{\sim} g^{\times} i^*$$

to be the natural isomorphism of [5, 4.3]. The transitivity properties of \mathbb{B}_{-} follow from the transitivity properties proved in [4, §1.8].

We have therefore shown that conditions (1) and (2) of 2.4.1 hold. Conditions (3) and (4) are easily verified just as in 7.1.1 above. Continuing along similar lines, (5) also follows from the arguments in 7.1.2 since the definition of the base-change isomorphisms are formally analogous.

As a result we now obtain the following output.

THEOREM 7.3.2. (1). *On the category \mathbf{Q}_{nn} of composites of étale maps and proper flat pseudo-coherent maps of quasi-compact quasi-separated schemes, there exists a pseudofunctor $(-)^!$, unique up to a unique isomorphism, taking values in $\mathbf{D}_{\text{qc}}^+(X)$ for any scheme X , such that the following conditions hold.*

- (i) *Over the subcategory of proper maps, $(-)^!$ gives the right adjoint to the derived direct-image pseudofunctor $(-)_*$.*
- (ii) *Over the subcategory of étale maps, $(-)^!$ is the inverse-image pseudofunctor $(-)^*$.*
- (iii) *For the fiber-product diagram \mathfrak{s} of (7.3.1.1), if i, j are étale, then the associated base-change isomorphism of (7.3.1.2) equals, via the identifications in (i) and (ii), the obvious pseudofunctorial isomorphism given by $(-)^!$.*

(2). *For the diagram \mathfrak{s} of (7.3.1.1), if f, g are in \mathbf{Q}_{nn} , then there is a flat-base-change isomorphism $\mathbb{B}_{\mathfrak{s}}^!$. Moreover $\mathbb{B}_{-}^!$ is horizontally and vertically transitive and is uniquely determined by the condition that when f, g are proper, $\mathbb{B}_{\mathfrak{s}}^!$ is the isomorphism of (7.3.1.2) while if f, g are étale, then $\mathbb{B}_{\mathfrak{s}}^!$ is the obvious isomorphism resulting from pseudofunctoriality of $(-)^*$.*

We remark here that the replacement of open immersions by étale maps in the above theorem is made possible by using essentially the same arguments as in the proof of 7.1.6.

7.4. A review of Huang's $(-)_{\#}$. We review Huang's construction of a pseudofunctor $(-)_{\#}$ in [3] in light of our pasting results. Since Huang's result concerns *covariant* pseudofunctors, so we work with the appropriate opposite category to obtain contravariance. We now show that in the situation of [3, Chp. 6], our input conditions [A]–[D] are attained.

Let \mathbf{C} be the opposite category of the category of residually finitely generated local homomorphisms of noetherian complete local rings. Let \mathbf{P} be the subcategory consisting of surjective homomorphisms and \mathbf{O} the subcategory consisting of formally smooth homomorphisms. For any (local) ring A in \mathbf{C} , let \mathcal{D}_A be the category of A -modules having zero-dimensional support.

For a map $T \rightarrow S$ in \mathbf{C} , the homomorphism of rings actually goes $S \rightarrow T$. In order to switch between these categories, we use an ordinary letter, say f for the ring homomorphism and the dotted letter \dot{f} for the corresponding \mathbf{C} -map. For a ring R in \mathbf{C} , m_R stands for its maximal ideal.

For $\dot{f}: T \rightarrow S$ in \mathbf{O} , we set $\dot{f}^\square := H_{m_T}^n(\omega_{T/S} \otimes_S -)$ where $n = \dim(T/m_S T)$ and $\omega_{T/S}$ is top exterior power of the universally complete module of relative differentials of T over S ([3, 3.10]). For $\dot{f}: T \rightarrow S$ in \mathbf{P} we set $\dot{f}^\times := \text{Hom}_S(T, -)$.

Let $T \xrightarrow{\dot{f}} S \xrightarrow{\dot{g}} R$ be maps in \mathbf{C} . If \dot{f}, \dot{g} are in \mathbf{P} , then the comparison map $C_{\dot{f}, \dot{g}}^\times$ is the one given by “evaluation at 1” while if \dot{f}, \dot{g} are in \mathbf{O} , then there are many natural candidates for $C_{\dot{f}, \dot{g}}^\square$. One possible candidate is given in [3, 2.5], which, for $M \in \mathcal{D}_R$ can be described in terms of generalized fractions as follows

$$(\dagger) \quad \left[\begin{array}{c} \beta \otimes \left[\begin{array}{c} \alpha \otimes m \\ \mathbf{s} \end{array} \right] \\ \mathbf{t} \end{array} \right] \longrightarrow \left[\begin{array}{c} (\beta \wedge \alpha) \otimes m \\ \mathbf{t}, \mathbf{s} \end{array} \right], \quad \{m \in M, \alpha \in \omega_{S/R}, \beta \in \omega_{T/S}\},$$

where \mathbf{s} is a system of parameters of $S/m_R S$ and \mathbf{t} of $T/m_S T$.

For the fibered product of a \mathbf{P} -map and an \mathbf{O} -map, the base-change isomorphism is given in [3, 3.6, 3.9]. The transitivity of this base-change is readily verified from its explicit description in terms of generalized fractions.

For \mathbf{C} -maps $R \xrightarrow{\dot{f}} S \xrightarrow{\dot{g}} R$ where \dot{f} is in \mathbf{P} and \dot{g} in \mathbf{O} and $gf = 1_R$, the fundamental isomorphism $\phi_{\dot{f}, \dot{g}}$ is obtained using residues. Indeed, for any $M \in \mathcal{D}_R$ and any regular system of parameters $\mathbf{s} = s_1, s_2, \dots, s_n$ of $S/m_R S$, the R -module $\dot{f}^\times \dot{g}^\square M$ can be canonically identified with the R -submodule of $\dot{g}^\square M$ consisting of elements written in terms of generalized fraction as $\left[\begin{array}{c} ds_1 \wedge ds_2 \wedge \dots \wedge ds_n \otimes m \\ s_1, s_2, \dots, s_n \end{array} \right]$ and $\phi_{\dot{f}, \dot{g}}$ sends this fraction to m . (see [3, Chp. 5].)

When it comes to verifying the compatibilities in §2.1[D](i), one runs into trouble. While [D](i)(a) and [D](ii) hold, [D](i)(b) fails. Using the descriptions of all the isomorphisms involved in terms of generalized fractions, the reader may verify that for any cartesian diagram of \mathbf{C} -maps as follows where $\dot{f} \in \mathbf{P}$, $\dot{g}, \dot{h} \in \mathbf{O}$,

$$\begin{array}{ccccc} R' & \xrightarrow{\dot{f}'} & S' & \xrightarrow{\dot{g}'} & R' \\ \downarrow \dot{h} & & \downarrow \dot{h}' & & \downarrow \dot{h} \\ R & \xrightarrow{\dot{f}} & S & \xrightarrow{\dot{g}} & R \end{array}$$

the commutativity of the diagram in [D](i)(b) is off by a factor of $(-1)^{ab}$ where $a = \dim(S/m_R S)$ and b is the transcendence degree of the residue field extension $k_R \rightarrow k_{R'}$.

The remedy lies in choosing the correct signed modification of $C_{-, -}^\square$ and this is what Huang does in [3, 6.10]. In the situation of (\dagger) , let us redefine $C_{\dot{f}, \dot{g}}^\square$ to be $(-1)^{cd}$ times the isomorphism given by (\dagger) where $c = \dim(S/m_R S)$ and d is the transcendence degree of $k_S \rightarrow k_T$. Then this new definition of $C_{-, -}^\square$ still gives a pseudofunctor on \mathbf{O} and now [D](i)(b) is satisfied. For the other compatibilities, the calculations do not change. Thus all the input conditions are attained.

Therefore the calculations listed above are all that are necessary to obtain a pseudofunctor on the whole of \mathbf{C} . Any such pseudofunctor is necessarily isomorphic to Huang’s $(-)^{\#}$ of [3, 6.12]. The canonicity of Huang’s construction comes from the limiting process he employs over various factorizations of a fixed map into a

O-map followed by a P-map. Here we should point out that the isomorphisms used in the limiting process in [3, Chp. 6] are special cases of our isomorphism $\Psi_{-, -}$ defined in 4.1.1.

7.5. Comparison with Deligne's result. In [1, pp. 303–318], Deligne gave an abstract criterion for pasting pseudofunctors on two subcategories into one on the whole category. Here we briefly compare his result with ours.

Deligne's input conditions are stated below as De-A to De-C, in reference to ours. For the purpose of this subsection, we assume that in our input conditions $\mathbf{C} = \mathbf{Q}$, the smallest subcategory containing \mathbf{O} and \mathbf{P} .

De-A. Same as [A] except that in place of [A](ii), we assume that every map f in \mathbf{C} admits a factorization $f = ip$ where $i \in \mathbf{O}$ and $p \in \mathbf{P}$.

De-B. Same as [B].

De-C. For every *commutative* square \mathfrak{s} as follows,

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ V & \xrightarrow{i} & Y \end{array}$$

there is an isomorphism $\beta_{\mathfrak{s}}^{\text{De}}: j^{\square} f^{\times} \xrightarrow{\sim} g^{\times} i^{\square}$ that is transitive in the usual way.

Deligne's pasting result says that under De-A, De-B and De-C, there is a pseudofunctor on \mathbf{C} which restricts to $(-)^{\times}$ on \mathbf{P} , $(-)^{\square}$ on \mathbf{O} , and that is compatible with β_{-}^{De} .

We remark here that there is also a dual variant. Consider the following condition.

De- $\check{\mathbf{A}}$. In De-A, assume that every f factors as $f = pi$ instead.

Then pasting also occurs for input data of De- $\check{\mathbf{A}}$, De-B and De-C. This is seen by interchanging the role of \mathbf{O} , $(-)^{\square}$ with \mathbf{P} , $(-)^{\times}$ in the original criterion.

Note that Deligne's result is incomparable with ours in that, neither De-A nor De- $\check{\mathbf{A}}$, implies or is implied by, our condition [A], even in the presence of the other conditions. Of course, De-C, trivially gives us [C]. We now give a sketch of how De-C alone also gives us [D].

For a sequence $X \xrightarrow{f} Y \xrightarrow{g} X$ where $f \in \mathbf{P}$, $g \in \mathbf{O}$ and $gf = 1_X$, the candidate for the fundamental isomorphism $f^{\times} g^{\square} \xrightarrow{\sim} 1_{\mathcal{D}_X}$ is $\beta_{\mathfrak{s}}^{\text{De}}$ corresponding to the following commutative square \mathfrak{s} .

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ f \downarrow & & \parallel \\ Y & \xrightarrow{g} & X \end{array}$$

For verifying [D](i), one takes a product of this diagram with $X' \rightarrow X$. Then one notes that cube lemma (4.2.5) also holds for a *commutative* cube of unit size with β_{-}^{De} in place of β_{-} , the proof being the same as the first half of the proof of 4.2.5, (see 4.3.1). Such a cube lemma together with an idempotence rule (see 3.2.5(i)) gives us [D](i). For [D](ii) one completes the diagram containing \mathfrak{s} there to a 2×2 diagram, the new maps all being identity on X . Now transitivity of β_{-}^{De} gives [D](ii).

ACKNOWLEDGEMENTS. I thank Joe Lipman and Pramath Sastry for constant encouragement and the stimulating discussions which led to this work. I am also grateful to them for sharing notes which have been helpful in the preparation of this paper.

References

- [1] P. Deligne, *Cohomologie à supports propres*, Théorie des topos et Cohomologie Étale des Schémas (SGA 4) Tome 3, Lecture Notes in Mathematics, vol. **305**, Springer-Verlag, New York, 1973, pp. 250–461.
- [2] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, **I**, Springer-Verlag, Heidelberg, 1971.
- [3] I. C. Huang, *Pseudofunctors on modules with zero dimensional support*, Memoirs, no. **548**, Amer. Math. Soc., 1995.
- [4] J. Lipman, *The residue theorem for formal schemes*, preprint, available at <http://www.math.purdue.edu/~lipman/>.
- [5] ———, *Non-noetherian Grothendieck duality*, Studies in Duality on Noetherian formal schemes and non-Noetherian ordinary schemes, Contemporary Mathematics, vol. **244**, Amer. Math. Soc. Providence RI, 1999, pp. 113–123.
- [6] J. Lipman, S. Nayak, and P. Sastry, *Pseudofunctorial behavior of Cousin complexes on formal schemes*, this volume.
- [7] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), 95–111.
- [8] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10.
- [9] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Duality and flat base change on formal schemes*, Studies in Duality on Noetherian formal schemes and non-Noetherian ordinary schemes, Contemporary Mathematics, vol. **244**, Amer. Math. Soc. Providence RI, 1999, pp. 1–87.
- [10] ———, *Correction to the paper “Duality and flat base change on formal schemes”*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 351–357.
- [11] J. L. Verdier, *Base change for twisted inverse image of coherent sheaves*, Algebraic Geometry, Oxford Univ. press, 1969, pp. 393–408.

CHENNAI MATHEMATICAL INSTITUTE, 92, G. N. CHETTY ROAD, CHENNAI-600017, INDIA
E-mail address: `snayak@cmi.ac.in`